Math 236 - Test 3
April 10, 2024

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

1. (5 points) Prove that for any $m \times n$ matrices G and H, $(G+H)^T = G^T + H^T$. (Helpful hint: The i, j entry of matrix M is the j, i entry of M^T.)

i,j - ENTRY OF $G+H$ IS $g_{i,j} + h_{i,j}$ $\Rightarrow$ i,j-ENTRY OF $(G+H)^T$ IS $g_{j,i} + h_{j,i}$

i,j-ENTRY OF G IS $g_{i,j}$ $\Rightarrow$ i,j-ENTRY OF G^T IS $g_{j,i}$

i,j-ENTRY OF H IS $h_{i,j}$ $\Rightarrow$ i,j-ENTRY OF H^T IS $h_{j,i}$

i,j-ENTRY OF $G^T + H^T$ IS $g_{j,i} + h_{j,i}$

SAME ENTRIES, SAME MATRIX!

2. (5 points) Recall that a matrix is symmetric if it is equal to its transpose. Use the result from problem 1 to prove that for any square matrix A, the matrix $A + A^T$ is symmetric. (Give a reason for each step of your proof.)

$$\underbrace{(A+A^T)^T = A^T + (A^T)^T}_{\text{PROBLEM 1}} \underbrace{= A^T + A}_{\text{SINCE } (A^T)^T = A} \underbrace{= A + A^T}_{\text{ADDITION IS COMMUTATIVE.}}$$

✓

3. (5 points) For $n \times n$ matrices S and T, should you expect $(S+T)(S-T) = S^2 - T^2$? Explain.

No, BECAUSE MATRIX MULT DOES NOT COMMUTE.

$$(S+T)(S-T) = S^2 \underbrace{- ST + TS}_{\text{NOT NECESSARILY } 0 \text{ MATRIX.}} - T^2$$

4. (5 points) Write the first product as a linear combination of the columns of the matrix and the second product as a linear combination of the rows.

$$\begin{pmatrix} 2 & 3 & 1 \\ -3 & 4 & 7 \\ -1 & 9 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \qquad \begin{pmatrix} 5 & -3 & -2 \end{pmatrix} \begin{pmatrix} 4 & -3 & 1 \\ 1 & 1 & -1 \\ 3 & -3 & 7 \end{pmatrix}$$

$$\boxed{2\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}}$$

$$5(4 \ -3 \ 1) - 3(1 \ 1 \ -1) - 2(3 \ -3 \ 7)$$

5. (5 points) For any square matrix A, the matrices A^TA and AA^T are symmetric. Prove it.

$(A^TA)^T = A^T(A^T)^T = A^TA \longleftarrow$ ITS TRANSPOSE IS EQUAL TO ITSELF. ✓

$(AA^T)^T = (A^T)^TA^T = AA^T \longleftarrow$ ITS TRANSPOSE IS EQUAL TO ITSELF. ✓

6. (8 points) Show that H has infinitely many right inverses, but that it has no left inverse.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

RIGHT INVERSE ...

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c & d \end{pmatrix} = I_2$$

$\Downarrow$

$c = 0$

$d = 1$

$\left.\begin{array}{l} a+e = 1 \\ b+f = 0 \end{array}\right\}$ INF MANY POSSIBILITIES

$$\begin{pmatrix} a & b \\ 0 & 1 \\ 1-a & -b \end{pmatrix}$$

LEFT INVERSE ..:

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b & a \\ c & d & c \\ e & f & e \end{pmatrix} = I_3$$

$\Downarrow$

$a = 1$ AND $a = 0$

NO SUCH a!

$\Downarrow$

NO SUCH INVERSE!

7. (6 points) Suppose G and H are invertible matrices. Is it true that $(G+H)^{-1} = G^{-1} + H^{-1}$? Justify your answer.

No way! $(G+H)(G^{-1}+H^{-1}) = I + GH^{-1} + HG^{-1} + I$ WHICH IS NOT NECESSARILY EQUAL TO I.

IN FACT, $G+H$ ISN'T EVEN NECESSARILY INVERTIBLE, e.g. $G = I$ & $H = -I$.

8. (6 points) Let D be an $n \times n$ diagonal matrix with diagonal entries $\delta_1, \delta_2, \ldots, \delta_n$.

(a) Give a condition on the δ's that completes the following phrase:

D is invertible if and only if $\delta_i \neq 0$ FOR $i = 1, 2, \ldots, n$.

Justify your answer.

$\det(D) = \delta_1 \cdot \delta_2 \cdots \delta_n \neq 0$ IFF ALL δ_i ARE NONZERO.

(b) Given your condition on the δ's, what is D^{-1}.

$$D^{-1} = \mathrm{diag}\left(\frac{1}{\delta_1}, \frac{1}{\delta_2}, \ldots, \frac{1}{\delta_n}\right) = \begin{pmatrix} 1/\delta_1 & & 0 \\ & 1/\delta_2 & \\ 0 & & \ddots & \\ & & & 1/\delta_n \end{pmatrix}$$

9. (6 points) Consider the matrix A, where x is a real number: $A = \begin{pmatrix} x & 2 \\ -4 & x \end{pmatrix}$.

(a) Show that A is nonsingular for any x.

$\det(A) = x^2 + 8$ WHICH IS NEVER ZERO FOR ANY REAL X.

(b) Compute A^{-1}.

$$A^{-1} = \frac{1}{x^2+8}\begin{pmatrix} x & -2 \\ 4 & x \end{pmatrix}$$

10. (12 points) Consider the bases $B, D \subseteq \mathcal{P}_2$.

$$B = \langle 1+x, x+x^2, 1+x^2 \rangle, \qquad D = \langle 2x, 1, 4x^2 \rangle$$

(a) Find the change-of-basis matrix with respect to B, D.

$$\left.\begin{aligned} \mathrm{Rep}_D(1+x) &= \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}_D \\ \mathrm{Rep}_D(x+x^2) &= \begin{pmatrix} 1/2 \\ 0 \\ 1/4 \end{pmatrix}_D \\ \mathrm{Rep}_D(1+x^2) &= \begin{pmatrix} 0 \\ 1 \\ 1/4 \end{pmatrix}_D \end{aligned}\right\} \text{By observation!}$$

$$\boxed{\mathrm{Rep}_{B,D}(id) = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/4 & 1/4 \end{pmatrix}_{B,D}}$$

(b) Let p be a polynomial with $\mathrm{Rep}_B(p) = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}_B$. Find $\mathrm{Rep}_D(p)$.

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 10 \\ 3 \end{pmatrix}_D}$$

11. (4 points) Is it possible for a change-of-basis matrix to be singular? Explain your reasoning.

No, the matrix must be nonsingular. This is a consequence of the fact that the identity is an isomorphism and that the bases are indeed bases. But it might be easier to think that the change from B to D affected by the matrix must be, able to be undone:

$$\mathrm{Rep}_{B,D}(id) \times \mathrm{Rep}_{D,B}(id) = I.$$

12. (12 points) Consider the basis for $\mathbb{R}^3$ shown below:

$$B = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\rangle.$$

Use the Gram-Schmidt process to find the corresponding orthonormal basis.

$$\vec{\delta}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{\delta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\vec{\delta}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{5/2}{2/4}\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5/2 \\ 5/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

ORTHOGONAL BASIS

$$D = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

NORMALIZE TO GET

$$\boxed{\left\langle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle}$$

13. (7 points) Suppose that $B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}$ is a set of ~~mutually~~ orthogonal vectors. Prove that the set is linearly independent.

NONZERO

Suppose $c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = \vec{0}$.

For $i = 1, 2, \ldots, n$,

$$\underbrace{\vec{\beta}_i \cdot \left(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n\right)} = \vec{\beta}_i \cdot \vec{0} = 0$$

$$c_1\vec{\beta}_1\cdot\vec{\beta}_i + c_2\vec{\beta}_2\cdot\vec{\beta}_i + \cdots + c_i\vec{\beta}_i\cdot\vec{\beta}_i + \cdots + c_n\vec{\beta}_n\cdot\vec{\beta}_i = 0$$

$$0 + 0 + \cdots + c_i\|\vec{\beta}_i\|^2 + 0 + \cdots + 0 = 0$$

$$c_i\|\vec{\beta}_i\|^2 = 0 \Rightarrow c_i = 0 \quad \checkmark$$

14. (4 points) Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

$I = AA^{-1}$

$$\Rightarrow \det(I) = \det(AA^{-1}) = \underbrace{\det(A)\det(A^{-1}) = 1}_{\det(A^{-1}) = \frac{1}{\det(A)}}$$

✓

15. (4 points) Compute the determinant of A by using a Laplace expansion.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$\det(A) = 3(0-6) - 0(0+2) + 1(3+2)$$

$$= -18 + 5 = \boxed{-13}$$

16. (6 points) Find the inverse by using the adjoint.

$$T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det(T) = 1(1) + 4(-1)$$
$$= -3$$

$$\operatorname{adj}(T) = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -4 \\ -3 & -3 & 9 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\boxed{T^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 0 & -4 \\ -3 & -3 & 9 \\ -1 & 0 & 1 \end{pmatrix}}$$