Math 236 - Test 3

April 10, 2024

Name ____________________

Score ____________

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

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1. (5 points) Prove that for any $m \times n$ matrices G and H, $(G+H)^T = G^T + H^T$. (Helpful hint: The i, j entry of matrix M is the j, i entry of M^T.)

2. (5 points) Recall that a matrix is symmetric if it is equal to its transpose. Use the result from problem 1 to prove that for any square matrix A, the matrix $A + A^T$ is symmetric. (Give a reason for each step of your proof.)

3. (5 points) For $n \times n$ matrices S and T, should you expect $(S+T)(S-T) = S^2 - T^2$? Explain.

4. (5 points) Write the first product as a linear combination of the columns of the matrix and the second product as a linear combination of the rows.

$$\begin{pmatrix} 2 & 3 & 1 \\ -3 & 4 & 7 \\ -1 & 9 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 5 & -3 & -2 \end{pmatrix} \begin{pmatrix} 4 & -3 & 1 \\ 1 & 1 & -1 \\ 3 & -3 & 7 \end{pmatrix}$$

5. (5 points) For any square matrix A, the matrices $A^T A$ and AA^T are symmetric. Prove it.

6. (8 points) Show that H has infinitely many right inverses, but that it has no left inverse.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

7. (6 points) Suppose G and H are invertible matrices. Is it true that $(G+H)^{-1} = G^{-1} + H^{-1}$? Justify your answer.

8. (6 points) Let D be an $n \times n$ diagonal matrix with diagonal entries $\delta_1, \delta_2, \ldots, \delta_n$.

   (a) Give a condition on the δ's that completes the following phrase:

   D is invertible if and only if ______________________________.

   Justify your answer.

   (b) Given your condition on the δ's, what is D^{-1}.

9. (6 points) Consider the matrix A, where x is a real number: $A = \begin{pmatrix} x & 2 \\ -4 & x \end{pmatrix}$.

   (a) Show that A is nonsingular for any x.

   (b) Compute A^{-1}.

10. (12 points) Consider the bases $B, D \subseteq \mathcal{P}_2$.

$$B = \langle 1+x, x+x^2, 1+x^2 \rangle, \qquad D = \langle 2x, 1, 4x^2 \rangle$$

(a) Find the change-of-basis matrix with respect to B, D.

(b) Let p be a polynomial with $\mathrm{Rep}_B(p) = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}_B$. Find $\mathrm{Rep}_D(p)$.

11. (4 points) Is it possible for a change-of-basis matrix to be singular? Explain your reasoning.

12. (12 points) Consider the basis for $\mathbb{R}^3$ shown below:

$$B = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\rangle.$$

Use the Gram-Schmidt process to find the corresponding orthonormal basis.

13. (7 points) Suppose that $B = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n\}$ is a set of nonzero mutually orthogonal vectors. Prove that the set is linearly independent.

14. (4 points) Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

15. (4 points) Compute the determinant of A by using a Laplace expansion.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{pmatrix}$$

16. (6 points) Find the inverse by using the adjoint.

$$T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$