Math 236 - Final Exam

May 8, 2024

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. Unless otherwise indicated, you may use your calculator to obtain any RREF.

1. (10 points) Consider the following system of linear equations.

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + 4z &= 0 \\ 5x + 6y + 7z &= 3 \end{aligned}$$

(a) Set up the corresponding augmented matrix and compute its RREF by hand. (You may use your calculator to check your answer.)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 3 \end{array}\right) \begin{array}{c} R_2 = -2R_1 + R_2 \\ R_3 = -5R_1 + R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -2 \end{array}\right) \; R_3 = -R_2 + R_3 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 = -R_2 + R_1 \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Use the RREF to determine the general solution of the system.

PARTICULAR SOLUTION IS $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

SOLUTION FOR CORRESPONDING HOMO IS $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

SOLUTION IS

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t, \quad t \in \mathbb{R}}$$

2. (4 points) Give an example of a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ that is not one-to-one and say why.

$$f(x,y) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \qquad f(1,1) = f(-1,-1)$$

BUT $(1,1) \neq (-1,-1)$.

3. (4 points) Show that U is a subspace of $\mathcal{M}_{2\times 2}$.

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

CLOSED UNDER LINEAR COMBOS.

$$\alpha \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha b_1 \\ 0 & \alpha c_1 \end{pmatrix} + \begin{pmatrix} \beta a_2 & \beta b_2 \\ 0 & \beta c_2 \end{pmatrix} = \begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ 0 & \alpha c_1 + \beta c_2 \end{pmatrix} \in U.$$

4. (2 points) Show that U_1 is NOT a subspace of $\mathcal{M}_{2\times 2}$.

$$U_1 = \left\{ \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} : b, c \in \mathbb{R} \right\}$$

U_1 DOES NOT CONTAIN THE ZERO VECTOR:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

5. (8 points) Determine whether the set is a linearly dependent or independent subset of $\mathcal{M}_{2\times 2}$. Then say whether or not it is a basis for $\mathcal{M}_{2\times 2}$.

$$\left\{ \begin{pmatrix} 0 & 3 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ -11 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} \right\}$$

NOT LIN. INDEP, BUT 1^{ST} THREE ARE!

NOT A BASIS.

$$\begin{pmatrix} 0 & 3 \\ 9 & 1 \end{pmatrix} c_1 + \begin{pmatrix} 8 & 1 \\ -11 & 1 \end{pmatrix} c_2 + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} c_3 + \begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} c_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 8c_2 + c_3 + c_4 &= 0 \\ 3c_1 + c_2 + c_3 + 2c_4 &= 0 \\ 9c_1 - 11c_2 + c_3 + 5c_4 &= 0 \\ c_1 + c_2 + c_3 - c_4 &= 0 \end{aligned} \longrightarrow \left(\begin{array}{cccc|c} 0 & 8 & 1 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 9 & -11 & 1 & 5 & 0 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Downarrow$

INF MANY c_1, c_2, c_3, c_4 MAKE THE ZERO MATRIX.

6. (4 points) Consider the basis $B = \langle 1+x, 1-x, x^2 \rangle$ for $\mathcal{P}_3$. Let $p(x) = 3 - x - 3x^2$ and find $\mathrm{Rep}_B(p(x))$.

$$a(1+x) + b(1-x) + cx^2 = 3 - x - 3x^2$$

$c = -3$

$a + b = 3$

$a - b = -1$

$a = 1$

$b = 2$

$$\mathrm{Rep}_B(p(x)) = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}_B$$

7. (10 points) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 & 7 \\ 1 & 0 & 1 & 1 & -1 \end{pmatrix}$.

RREF = $\begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

(a) Find a basis for the row space of A.

$\langle (1\ 0\ 1\ 0\ -1),\ (0\ 1\ 0\ 0\ 4),\ (0\ 0\ 0\ 1\ 0) \rangle$

(b) Find a basis for the column space of A.

1ST, 2ND, AND 4TH COLUMNS.

$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

(c) What is the rank of A. How do you know?

RANK OF A = ROW RANK = COLUMN RANK

= 3

THREE VECTORS IN EACH BASIS.

8. (4 points) Suppose A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What are the eigenvalues of A^3. Explain your reasoning.

$$Ax = \lambda x \Rightarrow A(Ax) = A(\lambda x) \Rightarrow A^2x = \lambda Ax = \lambda^2 x$$
$$\Rightarrow A(A^2x) = A^3x = A(\lambda^2 x) = \lambda^2 Ax = \lambda^3 x$$

$\boxed{\text{EIGENVALUES ARE } \lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots, \lambda_n^3.}$

9. (20 points) Consider the matrix

$$M = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}. \quad M - \lambda I = \begin{pmatrix} 5-\lambda & -10 & -5 \\ 2 & 14-\lambda & 2 \\ -4 & -8 & 6-\lambda \end{pmatrix}$$

(a) Find the characteristic polynomial of M. You need not simplify.

$$p(\lambda) = (5-\lambda)\left[(14-\lambda)(6-\lambda)+16\right] + 10\left[12-2\lambda+8\right] - 5\left[-16+56-4\lambda\right]$$

$$= (5-\lambda)(14-\lambda)(6-\lambda) - 16\lambda + 80$$

(b) If you factor the characteristic polynomial, you will find

$$p(\lambda) = (\lambda - 5)(\lambda - 10)^2.$$

What are the eigenvalues of M and their corresponding algebraic multiplicities?

$\lambda = 5$, mult 1 ; $\lambda = 10$, mult 2

(c) Find eigenvectors corresponding to the eigenvalues.

$\lambda = 5$

$$M - 5I = \begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -5/4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 5/4 \\ -1/2 \\ 1 \end{pmatrix} \quad \text{-or-} \quad \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

$\lambda = 10$

$$M - 10\lambda = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(d) What are the geometric multiplicities of the eigenvalues?

$\lambda = 5$, geo mult 1 ; $\lambda = 10$, geo mult 2

(e) Is M diagonalizable? If so, find matrices P and D so that $M = PDP^{-1}$, where D is a diagonal matrix.

$$P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

10. (6 points) Suppose that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent vectors in the inner product space V. Also suppose that $\vec{u}$ is a nonzero vector in V that is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$. Prove that $\vec{w}_1$, $\vec{w}_2$, and $\vec{u}$ are linearly independent.

S'pose
$c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{u} = \vec{0}.$

$\vec{u} \cdot (c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{u}) = 0.$

$0 + 0 + c_3\|\vec{u}\|^2 = 0$

$\Rightarrow c_3 = 0$

So now $c_1\vec{w}_1 + c_2\vec{w}_2 = \vec{0}$

WHICH MUST MEAN

$c_1 = c_2 = 0$ BECAUSE $\vec{w}_1$ & $\vec{w}_2$ ARE INDEP.

11. (4 points) Define a product on $\mathbb{R}^3$ by

$$\langle (x_1 \quad y_1 \quad z_1), (x_2 \quad y_2 \quad z_2) \rangle = x_1x_2 + y_1y_2.$$

Show that $\langle \cdot, \cdot \rangle$ is NOT an inner product by showing that one of the four inner product axioms fails.

LOOK AT $(0 \quad 0 \quad 3)$

$\langle (0 \quad 0 \quad 3), (0 \quad 0 \quad 3) \rangle = 0$ BUT $(0 \quad 0 \quad 3) \neq \vec{0}$

12. (6 points) Suppose A is a nonsingular matrix. Use induction to prove that $(A^p)^{-1} = (A^{-1})^p$ for any positive integer p.

① $p=1$: $(A^1)^{-1} = (A)^{-1} = A^{-1} = (A^{-1})^1$ TRUE FOR $p=1$.

② ASSUME TRUE FOR $p=n$ SO THAT

$(A^n)^{-1} = (A^{-1})^n.$

③ CONSIDER $(A^{n+1})^{-1}$. $(A^{n+1})^{-1} = (A^nA)^{-1} = A^{-1}\overbrace{(A^n)^{-1}}^{\text{INDUCTION HYPOTHESIS ②}} = A^{-1}(A^{-1})^n = (A^{-1})^{n+1}$ ✓

13. (3 points) Consider the vector space $\mathcal{P}_3$ with the inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\,dx.$$

Show that $p_1(x) = x$ and $p_2(x) = x^2 - \frac{3}{4}x$ are orthogonal.

$\int_0^1 (x^3 - \frac{3}{4}x^2)\,dx = \frac{1}{4}x^4 - \frac{1}{4}x^3 \Big|_0^1 = 0$ ✓

14. (10 points) Consider the homomorphism $h : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ defined by

$$h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + b + (c - d)x + dx^2.$$

(a) Before you work any other parts of this problem, determine the sum of the rank of h and the nullity of h, and say how you know.

$$\text{RANK OF } h + \text{NULLITY OF } h = \dim(\mathcal{M}_{2\times 2}) = \boxed{4}$$

(b) Find a basis for the range space of h. Then state the rank of h.

$$\{ a+b + cx + d(x^2-x) : a,b,c,d \in \mathbb{R} \}$$

$$= \{ e + cx + d(x^2-x) : e,c,d \in \mathbb{R}$$

$$= \text{span}(\{1, x, x^2-x\}) \qquad B = \langle 1, x, x^2-x \rangle$$

RANK $= 3$

(c) Find a basis for the null space of h. Then state the nullity of h.

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+(c-d)x+dx^2 = 0 \} \qquad \left.\begin{matrix} a+b=0 \\ c-d=0 \\ d=0 \end{matrix}\right\} \therefore \begin{matrix} a=-b \\ c=0 \\ d=0 \end{matrix}$$

$$= \{ \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{R} \}$$

$$= \text{span}(\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \})$$

NULLITY $= 1$

15. (5 points) Suppose A is an $n \times n$ matrix. Write five distinct statements that are equivalent to the statement A *is nonsingular*.

1. RREF$(A) = I_n$
2. ROWS OF A ARE LIN. INDEP.
3. RANK $A = n$
4. $A\vec{x} = \vec{0}$ IFF $\vec{x} = 0$
5. $\det(A) \neq 0$.
6. EIGENVALUES OF A ARE NONZERO.
7. A^{-1} EXISTS.