Math 236 - Assignment 3

February 4, 2026

Name KEY ______________

Score ______________

Show all work to receive full credit. Supply explanations when necessary. Do all computations by hand unless otherwise indicated. This assignment is due February 11.

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1. Let V be the set of all 2×2 nonsingular matrices with the usual operations of matrix addition and scalar multiplication. Show that V is NOT a vector space.

   Solution

   There are many ways that you could prove that V is not a vector space. I'll show that addition does not satisfy the closure property.

   The 2×2 matrices A and B are nonsingular (obviously).

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   However, the zero matrix, which is their sum, is singular.

2. Show that P is a vector space with the usual operations of polynomial addition and multiplication by a constant.

   $$P = \{p \in \mathcal{P}_2 : p(x) = p(-x) \text{ for all } x \in \mathbb{R}\}.$$

   (It might be helpful to start by determining a description for the polynomials in P.)

   Solution

   Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathcal{P}_2$. If $p \in P$, then

   $$ax^2 + bx + c = ax^2 - bx + c \text{ for all } x,$$

   and therefore, we must have $b = 0$. This is the defining condition for P, so it follows that

   $$P = \{ax^2 + c : a, c \in \mathbb{R}\}.$$

   Now let's verify the 10 vector space properties.

   Property 1: Take two arbitrary polynomials in P and add them:

   $$(a_1x^2 + c_1) + (a_2x^2 + c_2) = \cdots = (a_1 + a_2)x^2 + (c_1 + c_2),$$

   where we have used the commutative, associative, and distribute properties of operations on real numbers. The result is polynomial in P.

   Property 2: Refer to the addition shown above.

   $$(a_1x^2 + c_1) + (a_2x^2 + c_2) = \cdots = (a_1 + a_2)x^2 + (c_1 + c_2)$$

$$= (a_2 + a_1)x^2 + (c_2 + c_1) = \cdots = (a_2x^2 + c_2) + (a_1x^2 + c_1)$$

Property 3: Take three arbitrary polynomials in P, and consider the sum

$$[(a_1x^2 + c_1) + (a_2x^2 + c_2)] + (a_3x^2 + c_3).$$

Each two-term sum in parentheses simply represents some real number (once the variables take values). Since addition of real numbers is associative, we must have

$$(a_1x^2 + c_1) + [(a_2x^2 + c_2) + (a_3x^2 + c_3)].$$

Property 4: The polynomial $0x^2 + 0$ is a polynomial in P and

$$(ax^2 + c) + (0x^2 + 0) = ax^2 + c.$$

Therefore $0x^2 + 0$ is the "zero vector" in P.

Property 5: For any given polynomial in P, the polynomial with opposite coefficients is in P and works as the additive inverse in P:

$$(ax^2 + c) + (-ax^2 - c) = 0x^2 + 0.$$

Property 6: Take an arbitrary polynomial in P and multiply it by the scalar α:

$$\alpha\,(ax^2 + c) = \alpha(ax^2) + \alpha c = (\alpha a)x^2 + \alpha c.$$

The result is a polynomial in P.

Property 7: Take an arbitrary polynomial in P and multiply it by the sum of the scalars α and β:

$$(\alpha + \beta)\,(ax^2 + c).$$

Now expand and rewrite:

$$(\alpha + \beta)x^2 + (\alpha + \beta)c = \cdots = \alpha(ax^2 + c) + \beta(ax^2 + c),$$

where we have used the commutative, associative, and distribute properties of operations on real numbers.

Property 8: Take two arbitrary polynomials in P and the scalar α and consider.

$$\alpha\,[(a_1x^2 + c_1) + (a_2x^2 + c_2)].$$

Each two-term sum in parentheses simply represents some real number (once the variables take values). Using the distributive property of real-number multiplication over real-number addition, we get

$$\alpha\,[(a_1x^2 + c_1) + (a_2x^2 + c_2)] = \alpha\,(a_1x^2 + c_1) + \alpha\,(a_2x^2 + c_2).$$

Property 9: Take an arbitrary polynomial in P and multiply it by the product of the scalars α and β:

$$(\alpha\,\beta)\,(ax^2+c).$$

The two-term sum simply represents a real number, so using the associative property of real-number multiplication:

$$(\alpha\,\beta)\,(ax^2+c)=\alpha\,[\beta\,(ax^2+c)].$$

Property 10: Take an arbitrary polynomial in P and multiply by the scalar 1:

$$1\,(ax^2+c)=1ax^2+1c=ax^2+c.$$

3. Let V be the set of all vectors in $\mathbb{R}^2$ with the usual addition. However, define scalar multiplication '$\cdot$' in V as follows:

$$a\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax\\a^2y\end{pmatrix}.$$

Show that V is NOT a vector space.

Solution

Properties 1–5 all hold because those are "addition" properties. Let's find a property of scalar multiplication that fails.

For the vector $\begin{pmatrix}x\\y\end{pmatrix}$ in $\mathbb{R}^2$ and scalars a and b,

$$(a+b)\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}(a+b)x\\(a+b)^2y\end{pmatrix}=\begin{pmatrix}ax+bx\\a^2y+2aby+b^2y\end{pmatrix}\neq a\cdot\begin{pmatrix}x\\y\end{pmatrix}+b\cdot\begin{pmatrix}x\\y\end{pmatrix}.$$

4. Show that the set $\mathbb{R}^+$ of positive real numbers is a vector space when we interpret the "sum", $x+y$, as the product of x and y, and we interpret scalar "multiplication", $k\cdot x$, as the kth power of x.

Solution

Let's name the vector space V and verify that the 10 vector space properties hold in V.

Property 1: Take two positive real numbers x and y and "add" them: $x+y=xy$. Since the product of two positive real numbers is a positive real number, $x+y$ is in V.

Property 2: $x+y=y+x$ because multiplication of positive real numbers is commutative.

Property 3: $(x+y)+z=(xy)z=x(yz)=x+(y+z)$ because the multiplaction of positive real numbers is associative.

Property 4: The positive real number 1 is the zero vector in V: $x + 1 = x\,1 = 1x = x$.

Property 5: For any given positive real number, the positive real number $1/x$ works as the additive inverse in V: $(x + 1/x) = x\,(1/x) = 1$.

Property 6: Take an arbitrary positive real number in V and "multiply" it by the scalar α: $\alpha \cdot x = x^\alpha$. For any real number α, x^α is a positive real number. Therefore $\alpha \cdot x = x^\alpha$ is in V.

Property 7: Take an arbitrary positive real number in V and "multiply" it by the sum (the regular sum in $\mathbb{R}$) of the scalars α and β: $(\alpha+\beta)\cdot x = x^{\alpha+\beta} = x^\alpha x^\beta = (\alpha\cdot x)+(\beta\cdot x)$.

Property 8: Take two arbitrary positive real numbers in V and the scalar α: $\alpha\cdot(x+y) = (xy)^\alpha = x^\alpha y^\alpha = \alpha \cdot x + \alpha \cdot y$.

Property 9: Take an arbitrary positive real number in V and "multiply" it by the product (the regular product in $\mathbb{R}$) of the scalars α and β: $(\alpha\,\beta) \cdot x = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha = \alpha \cdot (\beta \cdot x)$.

Property 10: Take an arbitrary positive real number in V and "multiply" by the scalar 1: $1 \cdot x = x^1 = x$.

5. Prove that in a vector space, the zero vector is unique. Use only the ten vector space conditions. (Hint: We often prove the uniqueness of a mathematical object by assuming there are two objects with the given property, and then concluding that the objects must be the same.)

   Solution

   Suppose that $\vec{0}_1$ and $\vec{0}_2$ are zero vectors in the vector space V. Then

$$\vec{v} + \vec{0}_1 = \vec{v} \quad \text{and} \quad \vec{v} + \vec{0}_2 = \vec{v}, \text{ for all } \vec{v} \in V.$$

   In particular (using $\vec{v} = \vec{0}_2$ and $\vec{v} = \vec{0}_1$), we have

$$\vec{0}_2 + \vec{0}_1 = \vec{0}_2 \quad \text{and} \quad \vec{0}_1 + \vec{0}_2 = \vec{0}_1.$$

   It now follows from the commutative property of addition that $\vec{0}_1 = \vec{0}_2$.

6. Prove that in a vector space, if $a\vec{v} = \vec{0}$, then $a = 0$ or $\vec{v} = \vec{0}$. Use only the ten vector space conditions and/or Lemma 1.16. (Hint: You may use the fact that in any field of scalars, any nonzero scalar has a multiplicative inverse. If you need more of a hint, just ask.)

   Solution

   Assume that $a\vec{v} = \vec{0}$. If $a = 0$, there is nothing to prove. So let's assume $a \neq 0$. Then there is a real number a^{-1} with the property that $a^{-1}a = 1$. It follows that

$$a^{-1}(a\vec{v}) = a^{-1}\vec{0} = \vec{0}$$

and

$$a^{-1}(a\vec{v}) = (a^{-1}a)\vec{v} = 1\vec{v} = \vec{v}.$$

Therefore $\vec{v} = \vec{0}$.

7. Is $W = \{p \in \mathcal{P}_2 : p(1) = 1\}$ a subspace of $\mathcal{P}_2$?

Solution

No way! W does not contain the zero polynomial in $\mathcal{P}_2$.

8. Determine if $\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$ is in the span of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$. What about $\begin{pmatrix} -5 & 0 \\ -5 & -12 \end{pmatrix}$?

Solution

First, we look for constants a and b so that

$$a\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}.$$

There are no such constants because any linear combination of the two matrices will have a zero in the $(1, 2)$-position, not the required 1.

For the second question, the answer is yes. It is easy to verify that $a = 9$ and $b = -7$ do the trick:

$$9\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - 7\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ -5 & -12 \end{pmatrix}.$$

9. Show that R is a subspace of $\mathcal{P}_2$.

$$R = \{p \in \mathcal{P}_2 : p(2) = 0\}$$

Then parameterize the subspace's description, and express the subspace as a span of vectors in $\mathcal{P}_2$.

Solution

R is a subspace because it is closed under linear combinations: for any elements, $p(x)$ and $q(x)$, of R and any scalars, α and β,

$$\alpha \cdot p(2) + \beta \cdot q(2) = \alpha \cdot 0 + \beta \cdot 0 = 0.$$

Now rewrite

$$\begin{aligned} R = \{p \in \mathcal{P}_2 : p(2) = 0\} &= \{ax^2 + bx + c : 4a + 2b + c = 0\} \\ &= \{ax^2 + bx - (4a + 2b) : b, c \in \mathbb{R}\} \\ &= \operatorname{span}(\{x^2 - 4, x - 2\}). \end{aligned}$$