Math 236 - Assignment 4

February 18, 2026

Name ____________________

Score __________

Show all work to receive full credit. Supply explanations when necessary. You may use technology to solve any linear systems. This assignment is due February 25.

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1. Determine whether the set is a linearly dependent or independent subset of $\mathcal{M}_{2\times 2}$.

$$\left\{ \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 7 & -5 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \right\}$$

Solution

Let c_1, c_2, c_3, and c_4 be constants with

$$c_1 \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} + c_2 \begin{pmatrix} 7 & -5 \\ 3 & 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 & 5 \\ 2 & -3 \end{pmatrix} + c_4 \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$c_1+7c_2-c_3+3c_4 = 0$, $2c_1-5c_2+5c_3+c_4 = 0$, $3c_1+3c_2+2c_3+c_4 = 0$, $-2c_1+c_2-3c_3-c_4 = 0$.

Reduce the associated augmented matrix to RREF:

$$\begin{pmatrix} 1 & 7 & -1 & 3 & 0 \\ 2 & -5 & 5 & 1 & 0 \\ 3 & 3 & 2 & 1 & 0 \\ -2 & 1 & -3 & -1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices are not independent. In fact, $c_1 = 2$, $c_2 = -1$, $c_3 = -2$, and $c_4 = 1$ are the coefficients of a nonzero linear combination that give the zero matrix.

2. Determine whether the set is a linearly dependent or independent subset of $\mathcal{P}_2$.

$$\{2 + x + x^2, 1 - x + 2x^2, 3 + 2x - x^2\}$$

Solution

Let c_1, c_2, and c_3 be constants with

$$c_1(2 + x + x^2) + c_2(1 - x + 2x^2) + c_3(3 + 2x - x^2) = 0.$$

This equation is equivalent to the system

$$2c_1 + c_2 + 3c_3 = 0,\ c_1 - c_2 + 2c_3 = 0,\ c_1 + 2c_2 - c_3 = 0.$$

Reduce the associated augmented matrix to RREF:

$$\begin{pmatrix} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, $c_1 = c_2 = c_3 = 0$, and the set is linearly independent.

3. Suppose that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set. Prove that $\{\vec{u}, \vec{u}+2\vec{v}, \vec{u}+2\vec{v}+3\vec{w}\}$ is also a linearly independent set.

   Solution

   Let's look at

$$d_1\vec{u} + d_2(\vec{u}+2\vec{v}) + d_3(\vec{u}+2\vec{v}+3\vec{w}) = 0.$$

   Distribute and rearrange to get

$$(d_1+d_2+d_3)\vec{u} + (2d_2+2d_3)\vec{v} + 3d_3\vec{w} = 0.$$

   It follows from the linear independence of $\vec{u}$, $\vec{v}$, and $\vec{w}$ that

$$d_1+d_2+d_3 = 0, \quad 2d_2+2d_3 = 0, \quad 3d_3 = 0.$$

   The only solution is $d_1 = d_2 = d_3 = 0$.

4. Determine if the following set is a basis for $\mathcal{P}_2$?

$$\{3x^2 - x + 1, 5x - 1, 6x + 1\}$$

   Solution

   Let's first check linear independence:

$$c_1(3x^2 - x + 1) + c_2(5x - 1) + c_3(6x + 1) = 0 \Longrightarrow$$

$$3c_1 = 0, \; -c_1 + 5c_2 + 6c_3 = 0, \; c_1 - c_2 + c_3 = 0.$$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 5 & 6 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Yes! The "vectors" are linearly indpendent.

   Does the set span $\mathcal{P}_2$? Well, yes. We actually answered that with the work above. The coefficient matrix above is nonsingular, so there is a unique solution for any right-hand side, not just the zero vector.

   Another way to prove this is to just notice that the set contains 3 linearly independent vectors in a 3-dimensional space. They must form a basis for the space.

5. Find a basis for the vector space of 2×2 symmetric matrices. Then represent the matrix A with respect to your basis.

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

Solution

Let S be the vector space of 2×2 symmetric matrices. Then

$$S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

It should be pretty clear that a possible basis is

$$B = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

In terms of this basis,

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 9 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Or, we could write

$$\text{Rep}_B(A) = \begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix}_B.$$

6. Represent $p(x) = 2x - x^2 + 5x^3$ with respect to the given basis for $\mathcal{P}_3$.

$$B = \{1, 1 - x, 1 + x - x^2, 1 + x + x^2 - x^3\}$$

Solution

We must find c_1, c_2, c_3, and c_4 so that

$$c_1 + c_2(1 - x) + c_3(1 + x - x^2) + c_4(1 + x + x^2 - x^3) = 2x - x^2 + 5x^3.$$

It follows that

$$c_1 + c_2 + c_3 + c_4 = 0,$$
$$-c_2 + c_3 + c_4 = 2,$$
$$-c_3 + c_4 = -1,$$
$$-c_4 = 5.$$

Back solving, we find $c_1 = 20$, $c_2 = -11$, $c_3 = -4$, and $c_5 = -5$. Therefore

$$\text{Rep}_B(p(x)) = \begin{pmatrix} 20 \\ -11 \\ -4 \\ -5 \end{pmatrix}_B.$$

7. A matrix is called a *Toeplitz matrix* if its diagonal entries (descending from left to right) are constant. Find a basis for, and the dimension of, the vector space of 3×3 Toeplitz matrices.

Solution

Let T be the vector space of 3×3 Toeplitz matrices. Then

$$T = \left\{ \begin{pmatrix} a & b & c \\ d & a & b \\ e & d & a \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\}.$$

It should be pretty clear that a possible basis is

$$B = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

Up to order, B is the standard basis for T. The dimension of this vector space is 5.

8. Write M as the span of polynomials in $\mathcal{P}_3$. Then show that your polynomials are linearly independent. What is the dimension of M?

$$M = \{a + bx + cx^2 + dx^3 : 2a + b - c - 2d = 0\}$$

Solution

Use the condition to rewrite M:

$$\begin{aligned} M &= \{a + bx + (2a + b - 2d)x^2 + dx^3 : a, b, d \in \mathbb{R}\} \\ &= \{a(1 + 2x^2) + b(x + x^2) + d(-2x^2 + x^3) : a, b, d \in \mathbb{R}\}. \end{aligned}$$

Now let $B = \langle 1 + 2x^2, x + x^2, -2x^2 + x^3 \rangle$.

B is linearly independent?
Yes. Suppose

$$c_1(1 + 2x^2) + c_2(x + x^2) + c_3(-2x^2 + x^3) = 0.$$

Then, by equating coefficients, we have

$$c1 = 0,\ c_2 = 0,\ 2c_1 + c_2 - 2c_3 = 0,\ c_3 = 0.$$

It follows that $c_1 = c_2 = c_3 = 0$.

B is a linearly independent spanning set for M. Therefore it is a basis. The dimension of M is 3.

9. Find a basis for, and the dimension of, the solution set of the following system.

$$\begin{array}{rcrcrcrcl} x_1 & - & 4x_2 & + & 3x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 8x_2 & + & 6x_3 & - & 2x_4 & = & 0 \end{array}$$

Solution

The second equation is two times the first, so let's ignore it. With the remaining equation, we can backsolve, using x_2, x_3, and x_4 as free variables.

$$x_1 = 4x_2 - 3x_3 + x_4, \quad x_2 = x_2, \quad x_3 = x_3, \quad x_4 = x_4.$$

The solution space is

$$S = \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4 \right\}.$$

A basis for the solution space is

$$B = \left\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$\dim(S) = 3$.

10. Find a basis for the row space, a basis for the column space, and the rank of A.

$$A = \begin{pmatrix} 2 & 8 & -2 & -10 \\ -2 & -8 & 1 & 7 \\ -1 & -4 & 2 & 8 \\ 4 & 16 & -3 & -17 \end{pmatrix}$$

Solution

$$\text{rref}(A) = \begin{pmatrix} 1 & 4 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the RREF, a basis for the row space is

$$B_R = \langle \begin{pmatrix} 1 & 4 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 3 \end{pmatrix} \rangle,$$

a basis for the column space is

$$B_C = \left\langle \begin{pmatrix} 2 \\ -2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \\ -3 \end{pmatrix} \right\rangle.$$

and the rank is 2.