Math 236 - Assignment 5

February 25, 2026

Name ____________________

Score __________

Show all work to receive full credit. Supply explanations when necessary. You may use technology to solve any linear systems. This assignment is due March 4.

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1. Consider the matrix $M = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$.

   (a) Determine whether the row $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is in the row space of M.

   Solution

   The system

   $$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} a + \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} b + \begin{pmatrix} -1 & 2 & 7 \end{pmatrix} c$$

   gives

   $$-b - c = 1, \quad a + 2c = 1, \quad 3a + b + 7c = 1.$$

   The corresponding augmented matrix and its RREF are:

   $$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 7 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   No, $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is not in the row space of M. The last row of the RREF above shows a contradiction.

   (b) Find a basis for the row space of M.

   Solution

   $$\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

   From the RREF of M, we see that a basis for the row space is

   $$B_R = \langle \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \rangle.$$

   (c) Find the representation for the row $\begin{pmatrix} -3 & 8 & 27 \end{pmatrix}$ in terms of your basis.

   Solution

   The system

   $$\begin{pmatrix} -3 & 8 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} a + \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} b$$

   gives

   $$a = -3, \quad b = 8, \quad -a + 3b = 27.$$

   $$\mathrm{Rep}_{B_R}(\begin{pmatrix} -3 & 8 & 27 \end{pmatrix}) = \begin{pmatrix} -3 \\ 8 \end{pmatrix}_{B_R}.$$

2. Argue that the rank of a matrix is equal to the rank of its transpose.

   Solution

   The rows of A^T are the columns of A. Therefore the column rank of A is the row rank of A^T. Similarly, the row rank of A is the column rank of A^T. But since row and column rank are equal, all these ranks have to be the same.

3. Give an example to show that the column space of a matrix and the row space of the matrix are, in general, not the same even though they have the same dimension.

   Solution

   A very simple example is a 1×2 matrix such as $M = \begin{pmatrix} 1 & 2 \end{pmatrix}$. The row space is the set of all scalar multiples of $\begin{pmatrix} 1 & 2 \end{pmatrix}$, which is a subspace of $\mathcal{M}_{1\times 2}$. On the other hand, the column space is $\mathbb{R}$. Both have dimension 1, but they are distinctly different spaces (but they are isomorphic!).

4. Suppose that $A \in \mathcal{M}_{m\times n}$. Argue that the rank of A is less than or equal to $\min\{m, n\}$.
   Solution

   Since row rank and column rank are always equal, the rank cannot exceed the number of rows (m), and it also cannot exceed the number of columns (n). So $r \leq m$ and $r \leq n$. Therefore $r \leq \min\{m, n\}$.

5. Describe all matrices that have rank 0. Then find a general description for all matrices of rank 1.

   Solution

   The only $m \times n$ rank-zero matrix is the $m \times n$ zero matrix.

   A rank-one matrix must have at least one nonzero row (and column). Each nonzero row (or column) must be a scalar multiple of any nonzero row (or column).

6. Consider the function $F : \mathbb{R}^4 \to \mathcal{M}_{2\times 2}$ defined by

$$F(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) = \begin{pmatrix} c & a+d \\ b & d \end{pmatrix}.$$

   Show that F is one-to-one and onto.

   Solution

   Onto: Let $\begin{pmatrix} k & \ell \\ m & n \end{pmatrix}$ be an arbitrary element of $\mathcal{M}_{2\times 2}$. Then $(\ell - n, m, k, n)^T \in \mathbb{R}^4$

and

$$F\left(\begin{pmatrix} \ell - n \\ m \\ k \\ n \end{pmatrix}\right) = \begin{pmatrix} k & \ell \\ m & n \end{pmatrix}.$$

One-to-one: Suppose

$$F\left(\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}\right) = F\left(\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}\right).$$

Then

$$\begin{pmatrix} c_3 & c_1 + c_4 \\ c_2 & c_4 \end{pmatrix} = \begin{pmatrix} d_3 & d_1 + d_4 \\ d_2 & d_4 \end{pmatrix},$$

from which it follows that

$$c_3 = d_3, \quad c_1 + c_4 = d_1 + d_4, \quad c_2 = d_2, \quad c_4 = d_4.$$

From this, we get that

$$c_1 = d_1, \quad c_2 = d_2, \quad c_3 = d_3, \quad c_4 = d_4.$$

7. Define three different isomorphisms between $\mathbb{R}^3$ and $\mathcal{P}_2$. You don't need to prove that they are actually isomorphisms (just be sure of it).

<u>Solution</u>

These are pretty obvious isomorphisms...

$$F_1\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = ax^2 + bx + c, \qquad F_2\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = cx^2 + bx + a,$$

$$F_3\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = bx^2 + ax + c.$$

8. Explain why an isomorphism between $\mathbb{R}^3$ and $\mathcal{M}_{2\times 2}$ does not exist.

<u>Solution</u>

The vector spaces have dimensions 3 and 4, respectively. If there was an isomorphism from one onto the other, then the spaces would be isomorphic. But if they were isomorphormic, they would have the same dimension.

9. Show that the map $F : \mathcal{P}_2 \to \mathcal{P}_2$ given by $F(ax^2 + bx + c) = bx^2 - (a + c)x + a$ is an isomorphism.

Solution

Onto: Let $c_2x^2 + c_1x + c_0$ be an arbitrary element of $\mathcal{P}_2$. Then $c_0x^2 + c_2x + (-c_0 - c_1) \in \mathcal{P}_2$ and

$$F(c_0x^2 + c_2x + (-c_0 - c_1)) = c_2x^2 + c_1x + c_0.$$

One-to-one: Suppose $F(c_2x^2 + c_1x + c_0) = F(d_2x^2 + d_1x + d_0)$. Then

$$c_1x^2 - (c_2 + c_0)x + c_2 = d_1x^2 - (d_2 + d_0)x + d_2.$$

It follows that

$$c_1 = d_1 \qquad c_2 + c_0 = d_2 + d_0, \qquad c_2 = d_2,$$

from which it follows that

$$c_2 = d_2, \quad c_1 = d_1, \quad c_0 = d_0.$$

Linear: Let $p(x) = c_2x^2 + c_1x + c_0$ and $q(x) = d_2x^2 + d_1x + d_0$ be arbitrary elements of $\mathcal{P}_2$, and let $\alpha, \beta \in \mathbb{R}$.

$$F(\alpha p(x) + \beta q(x)) = \cdots = F((\alpha c_2 + \beta d_2)x^2 + (\alpha c_1 + \beta d_1)x + (\alpha c_0 + \beta d_0))$$
$$= (\alpha c_1 + \beta d_1)x^2 - [\alpha(c_2 + c_0) + \beta(d_2 + d_0)] + (\alpha c_2 + \beta d_2)$$
$$= \alpha[c_1x^2 - (c_2 + c_0)x + c_2] + \beta[d_1x^2 - (d_2 + d_0)x + d_2]$$
$$\alpha F(p(x)) + \beta F(q(x)).$$

10. For an arbitrary 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is defined by $\det(A) = ad - bc$. Show that the determinant function is not an isomorphism from $\mathcal{M}_{2\times 2}$ into $\mathbb{R}$.

Solution

The determinant function is not one-to-one. For example, let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Then $\det(P) = \det(Q) = 0$, but $P \neq Q$.

11. Suppose that $h : V \to W$ is a homomorphism and that $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is a linearly dependent set in V. Prove that $\{h(\vec{v_1}), h(\vec{v_2}), \ldots, h(\vec{v_n})\}$ is a linearly dependent set in W.

Solution

Since $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are dependent, at least one of the c's in the following equation is nonzero:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}_V.$$

Since h is a homomorphism,

$$h(c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}) =$$

$$c_1h(\vec{v_1}) + c_2h(\vec{v_2}) + \cdots + c_nh(\vec{v_n}) = h(\vec{0}_V) = \vec{0}_W.$$

Since at least one of the c's is nonzero, the vectors $h(\vec{v_1}), h(\vec{v_2}), \ldots, h(\vec{v_n})$ are linearly dependent.