# Math 236 - Assignment 7

March 25, 2026

Name ______________

Score __________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due April 1.

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1. Consider the homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 2z - x \\ x + 2y \end{pmatrix}.$$

Using

$$B = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \text{and} \quad D = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$$

as bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, find $\text{Rep}_{B,D}(h)$.

<u>Solution</u>

Some details are omitted.

$$\text{Rep}_{B,D}(h) = \begin{pmatrix} \text{Rep}_D(h\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}) & \text{Rep}_D(h\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}) & \text{Rep}_D(h\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}) \end{pmatrix}_{B,D}$$

$$= \begin{pmatrix} \text{Rep}_D(\begin{pmatrix} 1 \\ 5 \end{pmatrix}) & \text{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) & \text{Rep}_D(\begin{pmatrix} 4 \\ -2 \end{pmatrix}) \end{pmatrix}_{B,D} = \begin{pmatrix} -3 & 1 & 10 \\ 4 & 0 & -6 \end{pmatrix}_{B,D}$$

2. Consider the homomorphism $h : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ defined by

$$h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a + d) + bx - cx^2.$$

Using $B$ as the standard basis for $\mathcal{M}_{2\times 2}$ and

$$D = \langle 1, 1 - 2x, 1 + x + x^2 \rangle$$

as the basis for $\mathcal{P}_2$, find $\text{Rep}_{B,D}(h)$.

<u>Solution</u>

Some details are omitted.

$$\text{Rep}_{B,D}(h) = \begin{pmatrix} \text{Rep}_D(h\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) & \text{Rep}_D(h\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) & \text{Rep}_D(h\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) & \text{Rep}_D(h\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \end{pmatrix}_{B,D}$$

$$= \begin{pmatrix} \text{Rep}_D(1) & \text{Rep}_D(x) & \text{Rep}_D(-x^2) & \text{Rep}_D(1) \end{pmatrix}_{B,D} = \begin{pmatrix} 1 & 1/2 & 3/2 & 1 \\ 0 & -1/2 & -1/2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{B,D}$$

3. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be an arbitrary matrix in $\mathcal{M}_{2\times 2}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(\vec{x}) = A\vec{x}.$$

Prove that $f$ is a homomorphism.

Solution

Since matrix multiplication distributes over addition, we have

$$A(\alpha\vec{x} + \beta\vec{y}) = A(\alpha\vec{x}) + A(\beta\vec{y}).$$

So now it suffices to show that $A(c\vec{x}) = cA\vec{x}$ for any constant $c$.

$$A(c\vec{x}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} = \begin{pmatrix} a_{11}cx_1 + a_{12}cx_2 \\ a_{21}cx_1 + a_{22}cx_2 \end{pmatrix} = \begin{pmatrix} ca_{11}x_1 + ca_{12}x_2 \\ ca_{21}x_1 + ca_{22}x_2 \end{pmatrix}$$
$$= \begin{pmatrix} c(a_{11}x_1 + a_{12}x_2) \\ c(a_{21}x_1 + a_{22}x_2) \end{pmatrix} = c\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = cA\vec{x}$$

4. Determine the matrix representing the zero map from $\mathcal{P}_4$ to $\mathbb{R}^3$, with respect to the standard bases.

Solution

The matrix representing a map from $\mathcal{P}_4$ to $\mathbb{R}^3$ will be a $3 \times 5$ matrix. The matrix representing the zero map is $3 \times 5$ zero matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Write the following product as a linear combination of the columns of the matrix.

$$\begin{pmatrix} 2 & 4 & -5 \\ 0 & 8 & 6 \\ -1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 2 & 4 & -5 \\ 0 & 8 & 6 \\ -1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} + 5\begin{pmatrix} -5 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -8 \\ -16 \\ 8 \end{pmatrix} + \begin{pmatrix} -25 \\ 30 \\ 10 \end{pmatrix} = \begin{pmatrix} -27 \\ 14 \\ 15 \end{pmatrix}$$

6. Write the following product as a linear combination of the rows of the matrix.

$$\begin{pmatrix} 3 & 1 & -6 \end{pmatrix} \begin{pmatrix} 7 & 3 & 2 \\ 1 & 4 & -9 \\ 2 & -3 & 1 \end{pmatrix}$$

<u>Solution</u>

$$\begin{aligned} \begin{pmatrix} 3 & 1 & -6 \end{pmatrix} \begin{pmatrix} 7 & 3 & 2 \\ 1 & 4 & -9 \\ 2 & -3 & 1 \end{pmatrix} &= 3\begin{pmatrix} 7 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & -9 \end{pmatrix} - 6\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 21 & 9 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 4 & -9 \end{pmatrix} + \begin{pmatrix} -12 & 18 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 31 & -9 \end{pmatrix} \end{aligned}$$

7. Make up a $3 \times 3$ matrix of rank 3, and call it $A$. Then make up a $3 \times 3$ matrix of rank 2, and call it $B$. Compute $AB$ and find its rank.

<u>Solution</u>

Let

$$A = \begin{pmatrix} 2 & 3 & -7 \\ 1 & -2 & 5 \\ 0 & -3 & 2 \end{pmatrix}.$$

Since $\text{rref}(A) = I_3$, $A$ has rank 3.

Now let

$$B = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 4 & 7 \\ -2 & 0 & -5 \end{pmatrix}.$$

Rows 1 and 2 of $B$ are independent because they are not multiples of one another. Row 3 is the dfifference of Rows 1 and 2. Therefore, $B$ has rank 2.

The product $AB$ is given by

$$AB = \begin{pmatrix} 25 & 20 & 60 \\ -15 & -4 & -37 \\ -13 & -12 & -31 \end{pmatrix}.$$

Since

$$\text{rref}(AB) = \begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/8 \\ 0 & 0 & 0 \end{pmatrix},$$

$AB$ has rank 2.

8. A matrix is said to be *upper triangular* if all entries below the main diagonal are zero. That is, a matrix is upper triangular if its $i, j$ entry is zero whenever $i > j$. For

example,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

is upper triangular. Argue that the product of two $n \times n$ upper triangular matrices is an upper triangular matrix. (You need not give a formal proof, just a compelling argument.)

Solution

Suppose $U$ and $V$ are $n \times n$ upper triangular matrices. It follows that the $(i, j)$-entry of each is zero whenever $i > j$. Now let $W = UV$ and write

$$U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix},$$

where $U_k = k$th row of $U$. Also write

$$V = \begin{pmatrix} V_1 & V_2 & \cdots & V_n \end{pmatrix},$$

where $V_k = k$th column of $V$.

The $(i, j)$-entry of $W$ is

$$w_{i,j} = U_i \cdot V_j = \begin{pmatrix} 0 & \cdots & 0 & u_{i,i} & u_{i,i+1} & \cdots & u_{i,n} \end{pmatrix} \begin{pmatrix} v_{1,j} \\ v_{2,j} \\ \vdots \\ v_{j,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If $i > j$, then the $w_{i,j} = 0$. Therefore $W$ is upper tringular.

9. Find the inverse of $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$.

Solution

To find the inverse, we form the augmented matrix

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

and then we compute the RREF of the augmented matrix.

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & -1/5 & 1/10 & 2/5 \\ 0 & 1 & 0 & 2/5 & 3/10 & -4/5 \\ 0 & 0 & 1 & 0 & -1/2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{10} \begin{pmatrix} -2 & 1 & 4 \\ 4 & 3 & -8 \\ 0 & -5 & 10 \end{pmatrix}$$