

Math 236 - Test 1

February 11, 2026

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary. Unless otherwise indicated, you may use your calculator to obtain any RREF. Write all solution sets in our standard vector form.

1. (6 points) State the three elementary row operations.

- ① INTERCHANGE THE POSITIONS (SWAP) ANY TWO ROWS (EQUATIONS).
- ② MULTIPLY ANY ROW (EQUATION) BY A NONZERO CONSTANT.
- ③ ADD ANY MULTIPLE OF ONE ROW (EQUATION) TO ANOTHER.

2. (10 points) Reduce the system to echelon form (but not RREF). Then find the solution set by backsolving. Do not use your calculator for the reduction. Indicate which row operations you used.

$$\begin{aligned} 2x + 7y + 2z + w &= 21 \\ x + 2y + z - w &= 5 \\ 3y + 2w &= 7 \end{aligned}$$

$R_2 \leftrightarrow R_1$

$$\begin{aligned} x + 2y + z - w &= 5 \\ 2x + 7y + 2z + w &= 21 \\ 3y + 2w &= 7 \end{aligned}$$

$R_2 = -2R_1 + R_2$

$$\begin{aligned} x + 2y + z - w &= 5 \\ 3y + 3w &= 11 \\ 3y + 2w &= 7 \end{aligned}$$

$R_3 = -R_2 + R_3$

$$\begin{aligned} x + 2y + z - w &= 5 \\ 3y + 3w &= 11 \\ -w &= -4 \end{aligned}$$

$$\begin{aligned} w &= 4 \\ 3y &= 11 - 3(4) = -1 \\ y &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} x &= 5 + w - z - 2y \\ &= 5 + 4 - z + \frac{2}{3} \\ &= \frac{29}{3} - z \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 29/3 \\ -1/3 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z, \quad z \in \mathbb{R}$$

3. (4 points) The following system is in echelon form. State the leading variable(s) and the free variable(s).

$$\begin{array}{rcl} x & - 3z + 5w & = 1 \\ & 4y & = -9 \\ & & 6w = 3 \end{array}$$

x, y, w ARE THE LEADING VARIABLES.

z IS A FREE VARIABLE.

4. (6 points) Consider the linear system

$$\begin{array}{r} 2x - 4y = k \\ -x + ay = 5 \end{array}$$

where a and k are real numbers. Find all values of a and k for which the system has

- (a) no solution.

$a = 2$
 k CAN BE ANY NUMBER EXCEPT FOR -10

THIS WOULD MAKE ECHELON FORM

LOOK LIKE $2x - 4y = k$

$0 = 10 + k$

↑ MUST BE NONZERO

- (b) infinitely many solutions.

$a = 2$
 $k = -10$

THIS WOULD MAKE ECHELON FORM

LOOK LIKE

$2x - 4y = -10$

$0 = 0$

- (c) a unique solution.

$a \neq 2$
 k CAN BE ANY NUMBER

THIS WOULD MAKE ECHELON FORM

LOOK LIKE

$2x - 4y = k$

$(a-2)y = 5 + \frac{k}{2}$

↑ MUST BE NONZERO

↑ CAN BE ANY #

5. (8 points) Write the system as an augmented matrix. Then compute the RREF and determine the complete solution set. You may use your calculator to find the RREF.

$$\begin{aligned} 35x_1 + 7x_2 + 2x_3 - 50x_4 - 54x_5 &= 154 \\ 5x_1 + x_2 - 10x_4 - 9x_5 &= 30 \\ -10x_1 - 2x_2 + 25x_4 + 20x_5 &= -75 \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 35 & 7 & 2 & -50 & -54 & 154 \\ 5 & 1 & 0 & -10 & -9 & 30 \\ -10 & -2 & 0 & 25 & 20 & -75 \end{array} \right)$$

RREF →

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & & \text{rhs} \\ \hline 1 & 1/5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & & 2 \\ 0 & 0 & 0 & 1 & 2/5 & & -3 \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ -1/2 \\ -2/5 \\ 1 \end{pmatrix} x_5$$

$$x_2, x_5 \in \mathbb{R}$$

Leading: x_1, x_3, x_4 . Free: x_2, x_5

6. (8 points) According to our definition from class, what does it mean for a matrix to be nonsingular? Use our definition to show that the following matrix is singular. You may use your calculator as you see fit.

DEFINITION:

A MATRIX IS NONSINGULAR IF

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$$

IT IS SQUARE AND IT IS

THE COEFFICIENT MATRIX

OF A HOMOGENEOUS SYSTEM

WITH A UNIQUE SOLUTION.

THE MATRIX IS SQUARE
AND ITS CORRESPONDING
HOMO. SYSTEM HAS
AUGMENTED MATRIX

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & -1 & -1 & 0 \end{array} \right), \text{ WHICH}$$

REDUCES TO

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

THE HOMOGENEOUS
SYSTEM HAS INFINITE
SOLUTIONS.

CALL THIS \mathbb{R}^+

7. (10 points) On the set of positive real numbers, we will say $x \sim y$ if and only if $x - y$ is an integer. Prove that \sim is an equivalence relation.

REFLEXIVE:

LET $x \in \mathbb{R}^+$

$x - x = 0$, WHICH IS AN INTEGER.

∴ $x \sim x$. ✓

SYMMETRIC:

LET $x, y \in \mathbb{R}^+$

AND ASSUME $x \sim y$.

THEN $x - y = k \in \mathbb{Z}$.

IT FOLLOWS THAT

$$y - x = -k$$

AND $-k$ IS ALSO AN INTEGER.

∴ $y \sim x$. ✓

TRANSITIVE: LET $x, y, z \in \mathbb{R}^+$ AND ASSUME $x \sim y$ AND $y \sim z$.

THEN $x - y = k \in \mathbb{Z}$ AND $y - z = m \in \mathbb{Z}$.

IT FOLLOWS THAT

$$x - z = k + m \text{ AND}$$

$k + m$ IS ALSO AN INTEGER.

∴ $x \sim z$. ✓

8. (4 points) For real numbers a and b , we will say $a \sim b$ if and only if $|a - b| \leq 1$. Show that \sim is NOT an equivalence relation.

\sim IS REFLEXIVE & SYMMETRIC,
BUT IT IS NOT TRANSITIVE.

FOR EXAMPLE,

$$|1 - 2| \leq 1 \text{ AND } |2 - 3| \leq 1, \text{ BUT } |1 - 3| = 2, \text{ WHICH IS NOT LESS/EQUAL } 1.$$

9. (4 points) Give an example of a matrix that is row equivalent to A and say how you know.

OF COURSE, THE SIMPLEST EXAMPLE WOULD BE A ITSELF, BECAUSE "ROW EQUIVALENCE" IS A REFLEXIVE RELATION.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

BUT I'LL GO WITH $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{pmatrix}$,

WHICH FOLLOWS FROM A

$$\text{BY } R_2 = 2R_1 + R_2.$$

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2ND MATRIX FOLLOWS FROM A BY ROW OPS.

10. (5 points) Using any of the facts we know from class, explain why any 2×2 nonsingular matrix must be row equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ IS NONSINGULAR,

THEN THE SYSTEM $ax + by = 0$
 $cx + dy = 0$

HAS THE UNIQUE SOLUTION
 $x=0, y=0$. THEREFORE
THE SYSTEM'S RREF

MUST BE $\begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{pmatrix}$.

THEREFORE A REDUCES
TO $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ BY

ROW OPS, AND

A IS ROW EQUIV TO $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

N6

11. (4 points) A set W is closed under linear combinations if, for any scalars α and β and for any x and y in W , the linear combination $\alpha x + \beta y$ is also in W . Show that the following set is closed under linear combinations.

$$W = \{p(x) \in P_2 \mid p(x) = 0\}$$

Typo!

THAT WAS
MEANT TO
SAY $p(0) = 0$.

THE WAY THIS SITS, W

CONTAINS ON THE
ZERO POLY,

WHICH IS
OBVIOUSLY
CLOSED UNDER
LINEAR
COMBOS!

12. (5 points) Recall that in a vector space, the vectors x and y are called additive inverses if their sum is the zero vector. Every vector in a vector space must have an additive inverse. Prove that each vector in a vector space has a unique additive inverse.

LET \vec{u} BE ARBITRARY VECTOR IN V ,
AND SUPPOSE THAT BOTH \vec{a} & \vec{b}
ARE ITS ADDITIVE INVERSES.

THEN $\vec{u} + \vec{a} = \vec{0}$ AND $\vec{u} + \vec{b} = \vec{0}$,

SO THAT $\vec{u} + \vec{a} = \vec{u} + \vec{b}$.

NOW ADD \vec{a} TO BOTH SIDES.

$$\vec{a} + (\vec{u} + \vec{a}) = \vec{a} + (\vec{u} + \vec{b})$$

$$(\vec{a} + \vec{u}) + \vec{a} = (\vec{a} + \vec{u}) + \vec{b}$$

$$(\vec{u} + \vec{a}) + \vec{a} = (\vec{u} + \vec{a}) + \vec{b}$$

$$\vec{0} + \vec{a} = \vec{0} + \vec{b}$$

$$\vec{a} = \vec{b}. \quad \square$$

13. (10 points) Consider the set S of all 2×2 symmetric matrices with the usual matrix addition and scalar multiplication:

$$S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

S is a vector space. Prove any five of the vector space properties.

① CLOSURE UNDER ADDITION: $\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$, WHICH IS SYMMETRIC ✓

② COMMUTATIVE PROPERTY: $\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$
 $= \begin{pmatrix} d+a & e+b \\ e+b & f+c \end{pmatrix} = \begin{pmatrix} d & e \\ e & f \end{pmatrix} + \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ ✓

③ ADDITIVE INVERSE: $\begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$ IS SYMMETRIC,
 AND $\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

④ CLOSURE UNDER MULT: $k \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} ka & kb \\ kb & kc \end{pmatrix}$, WHICH IS SYMMETRIC. ✓
 WHICH IS THE 0 VEC. IN S . ✓

⑤ MULTIPLICATIVE IDENTITY: $1 \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1a & 1b \\ 1b & 1c \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ ✓

14. (4 points) Look back at the problem above, but now let N be the set of all 2×2 nonsymmetric matrices (with the usual operations). Show that N is NOT closed under addition.

WITH a, b, c, d ALL UNEQUAL,
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ AND $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$
 ARE NOT SYMMETRIC
 (IN N).

THERE SUM IS $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,
 WHICH IS SYMMETRIC
 AND NOT IN N .

$$a = -2b - c$$

$$d = -b - c$$

$$P = \{ (-2b-c)x^3 + bx^2 + cx + (-b-c) : b, c \in \mathbb{R} \}$$

15. (12 points) Consider the following set P as a subset of \mathcal{P}_3 with the usual operations of polynomial addition and multiplication by a constant.

$$P = \{ ax^3 + bx^2 + cx + d : a + 2b + c = 0 \text{ and } b + c + d = 0 \}$$

- (a) Show that P is closed under scalar multiplication.

Let $p(x) = (-2b-c)x^3 + bx^2 + cx + (-b-c)$ where b, c are real numbers.

$$\begin{aligned} \alpha p(x) &= \alpha(-2b-c)x^3 + \alpha bx^2 + \alpha cx + \alpha(-b-c) \\ &= \dots = [-2(\alpha b) - (\alpha c)]x^3 + (\alpha b)x^2 + (\alpha c)x + [-(\alpha b) - (\alpha c)] \\ &= (-2k-m)x^3 + kx^2 + mx + (-k-m) \in P \quad \checkmark \end{aligned}$$

- (b) Show that any element of P has an additive inverse in P .

For any $p(x) = (-2b-c)x^3 + bx^2 + cx + (-b-c)$ be some arbitrary poly. in P .

Then $-p(x) = (2b+c)x^3 - bx^2 - cx + (b+c)$ is also a poly in P because it has the correct form.

$$\begin{aligned} p(x) + (-p(x)) &= 0x^3 + 0x^2 + 0x + 0 \\ &= \text{ZERO VECTOR IN } P. \quad \checkmark \end{aligned}$$

- (c) Write P as the span of two polynomials in \mathcal{P}_3 .

SEE ABOVE! $P = \{ b(-2x^3 + x^2 - 1) + c(-x^3 + x - 1) : b, c \in \mathbb{R} \}$

$$= \text{span} \left(\{ -2x^3 + x^2 - 1, -x^3 + x - 1 \} \right)$$

- (d) Is P a subspace of \mathcal{P}_3 ? How do you know?

Yes! P is the span of two polys in \mathcal{P}_3 . The span of any subset is a subspace.