Math 236 - Final Exam

May 13, 2026

Name key

Score ______

Show all work to receive full credit. Supply explanations when necessary. Unless otherwise indicated, you may use your calculator to obtain any RREF.

1. (12 points) Write the system as an augmented matrix and then reduce it to RREF by hand. Indicate which row operations you used. Write the solution set in vector form.

$$\begin{array}{rcrcrcrcr} x & + & 3y & + & z & + & w & = & 6 \\ & - & y & & & + & 2w & = & -4 \\ 2x & + & 5y & + & 2z & + & 4w & = & 8 \\ x & & & & & + & 7w & = & -6 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 6 \\ 0 & -1 & 0 & 2 & -4 \\ 2 & 5 & 2 & 4 & 8 \\ 1 & 0 & 0 & 7 & -6 \end{array}\right) \xrightarrow[R_4 = R_4 - R_1]{R_3 = R_3 - 2R_1} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 6 \\ 0 & -1 & 0 & 2 & -4 \\ 0 & -1 & 0 & 2 & -4 \\ 0 & -3 & -1 & 6 & -12 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 = R_3 + R_2 \\ R_4 = R_4 + 3R_2}]{R_2 = -R_2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 6 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right) \xrightarrow[R_3 = -R_3]{R_4 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 6 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 - 3R_2]{R_1 = R_1 - R_3} \begin{array}{c} \begin{array}{cccc|c} x & y & z & w & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 7 & -6 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

THIS IS RREF

w IS FREE

$$\boxed{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \\ 0 \\ 1 \end{pmatrix} w, \quad w \in \mathbb{R}}$$

2. (12 points) Consider the homomorphism $h : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ defined by

$$h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + b + c + dx^2.$$

(a) Before you work any other parts of this problem, determine the sum of the rank of h and the nullity of h, and say how you know.

$\text{rank}(h) + \text{nullity}(h) = \boxed{4}$

DIMENSION OF DOMAIN.

$\dim(\mathcal{M}_{2\times 2}) = 4$

(b) Find a basis for the range space of h. Then state the rank of h.

$R(h) = \{ a+b+c+dx^2 : a,b,c,d \in \mathbb{R} \}$

$= \{ e + dx^2 : e, d \in \mathbb{R} \}$

$= \text{span}(\{1, x^2\})$

$\boxed{\text{BASIS} = \langle 1, x^2 \rangle, \quad \text{rank}(h) = 2}$

(c) Find a basis for the null space of h. Then state the nullity of h.

$N(h) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c = 0 \text{ AND } d = 0 \right\}$

$= \left\{ \begin{pmatrix} a & b \\ -a-b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$= \text{span}\left(\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right)$

$\boxed{\text{BASIS} = \left\langle \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \quad \text{nullity}(h) = 2}$

3. (15 points) Consider the vector space $\mathcal{M}_{2\times 2}$ with bases B and D:

$$B = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \rangle,$$

$$D = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \rangle.$$

(a) Find the change-of-basis matrix with respect to B, D.

$$\mathrm{Rep}_{B,D}(id) = \left(\mathrm{Rep}_D(\vec{\beta}_1) \;\middle|\; \mathrm{Rep}_D(\vec{\beta}_2) \;\middle|\; \mathrm{Rep}_D(\vec{\beta}_3) \;\middle|\; \mathrm{Rep}_D(\vec{\beta}_4) \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{B,D}$$

EACH COLUMN IS EASY TO COMPUTE!

(b) Let $A = \begin{pmatrix} 5 & 2 \\ -3 & 7 \end{pmatrix}$. Find $\mathrm{Rep}_B(A)$.

$$a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -3 & 7 \end{pmatrix}$$

$$\begin{aligned} -b + c &= 5 \\ a &= 2 \\ a - d &= -3 \\ c &= 7 \end{aligned}$$

$a = 2,\ c = 7,\ d = 5,\ b = 2$

$$\boxed{\mathrm{Rep}_B(A) = \begin{pmatrix} 2 \\ 2 \\ 7 \\ 5 \end{pmatrix}_B}$$

(c) Use your change-of-basis matrix to find $\mathrm{Rep}_D(A)$.

$$\mathrm{Rep}_D(A) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{B,D} \begin{pmatrix} 2 \\ 2 \\ 7 \\ 5 \end{pmatrix}_B$$

$$= \boxed{\begin{pmatrix} -3 \\ -2 \\ 5 \\ -7 \end{pmatrix}_D}$$

CHECK: $\begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -3 & 7 \end{pmatrix}$ ✓

4. (20 points) Consider the matrix

$$M = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of M.

$$\det(M-\lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\left[(3-\lambda)^2-4\right] = (5-\lambda)(\lambda^2-6\lambda+5)$$
$$= (5-\lambda)(\lambda-5)(\lambda-1).$$

$\underline{\lambda=5}$ $\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\vec{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2 FREE VARIABLES

$\underline{\lambda=1}$ $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

1 FREE VARIABLE

$\boxed{\lambda=5;\ \vec{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$ $\boxed{\lambda=1;\ \vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

(b) What are the algebraic and geometric multiplicities of the eigenvalues?

$\boxed{\lambda=5 \rightarrow \text{Alg} = \text{Geo} = 2}$ $\boxed{\lambda=1 \rightarrow \text{Alg} = \text{Geo} = 1}$

(c) Is M diagonalizable? If so, find matrices P and D so that $M = PDP^{-1}$, where D is a diagonal matrix.

$\boxed{M \text{ IS DIAGONALIZABLE!}}$ $\boxed{P = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

5. (13 points) For any two polynomials, $p(x) = a + bx + cx^2$ and $q(x) = \alpha + \beta x + \gamma x^2$ in $\mathcal{P}_2$, define

$$\langle p, q\rangle = 4\alpha a + 2\beta b + \gamma c.$$

(a) $\langle\cdot,\cdot\rangle$ is an inner product. Prove any three of the four fundamental properties of the inner product.

1. $\langle q, p\rangle = 4a\alpha + 2b\beta + c\gamma = 4\alpha a + 2\beta b + \gamma c = \langle p, q\rangle$

2. $\langle p, p\rangle = 4a^2 + 2b^2 + c^2 \geq 0$ BECAUSE EACH TERM ≥ 0

3. NOW SUPPOSE $\langle p, p\rangle = 0$. THEN $4a^2 + 2b^2 + c^2 = 0$

THIS HAPPENS IFF $a = b = c = 0$,

I.E., WHEN $p(x) = 0$

(b) Find a nonzero polynomial in $\mathcal{P}_2$ that is orthogonal to $f(x) = 3 - 2x + 6x^2$ with respect to this inner product.

LET $p(x) = a + bx + cx^2$

$0 = \langle p, f\rangle = 12a - 4b + 6c$

CHOOSE $a = 2$, $b = 3$, $c = -2$

$\boxed{p(x) = 2 + 3x - 2x^2}$

(c) True or false? The standard basis on $\mathcal{P}_2$ is an orthonormal basis with respect to this inner product.

$\langle 1, x, x^2\rangle$ FALSE. THE "VECTORS" ARE ORTHOGONAL BUT NOT NORMALIZED

$\langle 1, 1\rangle = 4 \neq 1$

ORTHONORMAL:

$\langle \frac{1}{2}, \frac{1}{\sqrt{2}}x, x^2\rangle$

6. (7 points) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Use induction to prove that for any positive integer n,

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

THE CONCLUSION IS DEFINITELY TRUE WHEN $n=1$ SINCE $A = A$. ✓

NOW SUPPOSE THE CONCLUSION IS TRUE FOR $n = k$:

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

IT FOLLOWS THAT $A^{k+1} = A^k A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+k \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$. ✓

BY INDUCTION, THE CONCLUSION IS TRUE FOR ALL $n = 1, 2, 3, \ldots$

7. (6 points) Suppose that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent vectors in the inner product space V. Also suppose that $\vec{u}$ is a nonzero vector in V that is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$. Prove that $\vec{w}_1$, $\vec{w}_2$, and $\vec{u}$ are linearly independent.

SUPPOSE $c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{u} = \vec{0}$.

NOW FORM THE INNER PRODUCT

$$0 = \langle \vec{0}, \vec{u} \rangle = \langle c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{u}, \vec{u} \rangle$$

$$= c_1\langle \vec{w}_1, \vec{u} \rangle + c_2\langle \vec{w}_2, \vec{u} \rangle + c_3\langle \vec{u}, \vec{u} \rangle$$

$$= 0 + 0 + c_3\|\vec{u}\|^2$$

SINCE $\vec{u} \neq \vec{0}$, WE MUST HAVE $\underline{c_3 = 0}$.

SO NOW WE HAVE

$$c_1\vec{w}_1 + c_2\vec{w}_2 = \vec{0},$$

AND SINCE $\vec{w}_1$ & $\vec{w}_2$ ARE INDEP.,

$\underline{c_1 = c_2 = 0}$.

$\therefore$ $c_1 = c_2 = c_3 = 0$.

8. (8 points) Find a basis for the row space, a basis for the column space, and the rank of the matrix A.

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 13/11 \\ 0 & 1 & 0 & -17/11 \\ 0 & 0 & 1 & 6/11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Row space basis} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 13/11 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -17/11 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 6/11 \end{pmatrix}^T \right\rangle$$

$$\text{Column space basis} = \left\langle \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\rangle$$

$$\text{rank}(A) = 3$$

9. (7 points) Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

Suppose λ is an eigenvalue of A with eigenvector $\vec{x} \neq \vec{0}$.

Then $A\vec{x} = \lambda\vec{x}$ and $\vec{0} = A^2\vec{x} = A(A\vec{x}) = A\lambda\vec{x}$

$= \lambda A\vec{x} = \lambda^2\vec{x}$.

Since $\vec{x} \neq \vec{0}$ yet $\lambda^2\vec{x} = \vec{0}$,

we must have $\lambda^2 = 0$

or

$\lambda = 0$.