Math 240 - Quiz 3 Name key

September 24, 2020 Score ______

Show all work to receive full credit. Supply explanations when necessary. This quiz is due on September 29.

1. (3 points) Consider the following ODE:

$$x^2y'' + 2xy' - 6y = 0.$$

(a) Show that $y_1(x) = x^2$ and $y_2(x) = x^{-3}$ are solutions for $x > 0$.

$y_1'(x) = 2x$

$y_1''(x) = 2$

$x^2(2) + 2x(2x) - 6x^2$

$= 6x^2 - 6x^2 = 0$ ✓

$y_2'(x) = -3x^{-4}$

$y_2''(x) = 12x^{-5}$

$x^2(12x^{-5}) + 2x(-3x^{-4})$

$- 6x^{-3} = 12x^{-3} - 6x^{-3} - 6x^{-3} = 0$ ✓

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -\frac{3}{x^2} - \frac{2}{x^2} = -\frac{5}{x^2} \neq 0 \text{ on } (0,\infty).$$

(c) Find the unique solution that satisfies $y(2) = 10$ and $y'(2) = 15$.

$y(x) = c_1x^2 + c_2x^{-3}$, $y'(x) = 2c_1x - 3c_2x^{-4}$

$y(2) = 10 \Rightarrow 4c_1 + \frac{1}{8}c_2 = 10$

$y'(2) = 15 \Rightarrow 4c_1 - \frac{3}{16}c_2 = 15$

$\frac{5}{16}c_2 = -5 \Rightarrow c_2 = -16$

$c_1 = 3$

$\boxed{y(x) = 3x^2 - 16x^{-3}}$

2. (3 points) Consider the following ODE:

$$yy'' + (y')^2 = 0.$$

(a) Show that $y_1(x) = 1$ and $y_2(x) = \sqrt{x}$ are solutions.

$y_1'(x) = 0$

$y_1''(x) = 0$

$(1)(0) + (0)^2 = 0$ ✓

$y_2'(x) = \frac{1}{2}x^{-1/2}$

$y_2''(x) = -\frac{1}{4}x^{-3/2}$

$x^{1/2}\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2$

$= -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0$ ✓

Turn over.

(b) Show that $y_1(x) + y_2(x)$ is not a solution.

$y = 1 + \sqrt{x}$

$y' = \frac{1}{2}x^{-1/2}$

$y'' = -\frac{1}{4}x^{-3/2}$

$$yy'' + (y')^2 = (1 + x^{1/2})\left(-\frac{1}{4}x^{-3/2}\right) + \frac{1}{4}x^{-1}$$

$$= -\frac{1}{4}x^{-3/2} - \frac{1}{4}x^{-1} + \frac{1}{4}x^{-1}$$

$$= -\frac{1}{4}x^{-3/2} \neq 0$$

NOT A SOLUTION!

(c) Why should you not expect the sum of solutions to be a solution?

LINEAR COMBINATIONS OF SOLUTIONS ARE SOLUTIONS FOR LINEAR EQUATIONS. THIS EQUATION IS NOT LINEAR --- WE SHOULD NOT EXPECT A LINEAR COMBINATION OF SOLUTIONS TO BE A SOLUTION.

3. (4 points) Find the general solution.

(a) $4y'' + 8y' + 3y = 0$

$4r^2 + 8r + 3 = 0$

$(2r+1)(2r+3) = 0$

$r = -\frac{1}{2}, \quad r = -\frac{3}{2}$

$y_1 = e^{-x/2}$

$y_2 = e^{-3x/2}$

$$\boxed{y(x) = C_1 e^{-x/2} + C_2 e^{-3x/2}}$$

(b) $9y'' - 12y' + 4y = 0$

$9r^2 - 12r + 4 = 0$

$(3r-2)(3r-2) = 0$

$r = \frac{2}{3}$ (MULTIPLICITY 2)

$y_1 = e^{2x/3}$

$y_2 = xe^{2x/3}$

$$\boxed{y(x) = C_1 e^{2x/3} + C_2 x e^{2x/3}}$$