Math 240 - Quiz 5
October 8, 2020

Name key

Score

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. You must work individually on this quiz. This quiz is due October 13.

1. (2 points) Find by inspection particular solutions of

$$y'' + 2y = 4 \qquad \text{and} \qquad y'' + 2y = 6x.$$

Then use your results to find the general solution of $y'' + 2y = 6x + 4$.

$y'' + 2y = 4$

$\Downarrow$

$y_{p_1}(x) = 2$

$y'' + 2y = 6x$

$\Downarrow$

$y_{p_2}(x) = 3x$

Homogeneous eq'n: $y'' + 2y = 0$

Char eq'n: $r^2 + 2 = 0$

$r = \pm\sqrt{2}\,i$

$y_c(x) = C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x$

$y(x) = C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x + 2 + 3x$

2. (2 points) Solve the initial value problem.

$$y^{(3)} + 10y'' + 25y' = 0; \quad y(0) = 3,\ y'(0) = 4,\ y''(0) = 5$$

Char eq'n: $r^3 + 10r^2 + 25r = 0$

$r(r+5)(r+5) = 0$

$r = 0,\ r = -5,\ r = -5$

$y(x) = C_1 + C_2 e^{-5x} + C_3 x e^{-5x}$

$y(0) = 3 \Rightarrow C_1 + C_2 = 3$

$y'(x) = -5C_2 e^{-5x} + C_3 e^{-5x} - 5C_3 x e^{-5x}$

$y'(0) = 4 \Rightarrow -5C_2 + C_3 = 4$

$y''(x) = 25C_2 e^{-5x} - 5C_3 e^{-5x} - 5C_3 e^{-5x} + 25C_3 x e^{-5x}$

$y''(0) = 5 \Rightarrow 25C_2 - 10C_3 = 5$

$25C_2 - 10C_3 = 5$

$-50C_2 + 10C_3 = 40$

$-25C_2 = 45$

$C_2 = -9/5$

$C_3 = -5$

$C_1 = 24/5$

$y(x) = \frac{24}{5} - \frac{9}{5} e^{-5x} - 5x e^{-5x}$

3. (2 points) Solve. $y^{(3)} + 3y'' + 4y' - 8y = 0$

CHAR EQ'N : $r^3 + 3r^2 + 4r - 8 = 0$

By OBSERVATION, $r = 1$ IS A SOLUTION,

SO $r-1$ IS A FACTOR

$$\begin{array}{r|l} & r^2+4r+8 \\ r-1 & r^3+3r^2+4r-8 \\ & -(r^3-r^2) \\ \hline & 4r^2+4r \\ & -(4r^2-4r) \\ \hline & 8r-8 \\ & 8r-8 \\ \hline & 0 \end{array}$$

$(r-1)(r^2+4r+8) = 0$

$r = 1,\ r = \dfrac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i$

$y(x) = c_1 e^{x} + c_2 e^{-2x} \cos 2x + c_3 e^{-2x} \sin 2x$

4. (2 points) The equation of motion of a mass in a mass-spring system satisfies the equation

$$4x'' + 20x' + 169x = 0; \quad x(0) = 4,\ x'(0) = 16.$$

Determine the equation of motion. Write your final answer in terms of a single sine or cosine.

CHAR EQ'N : $4r^2 + 20r + 169 = 0$

$r = \dfrac{-20 \pm \sqrt{400 - 16(169)}}{8}$

$= \dfrac{-20 \pm \sqrt{2304}\, i}{8}$

$= \dfrac{-20 \pm 48i}{8} = -\frac{5}{2} \pm 6i$

$x(t) = c_1 e^{-\frac{5}{2}t} \cos 6t + c_2 e^{-\frac{5}{2}t} \sin 6t$

$x(0) = 4 \Rightarrow c_1 = 4$

$x'(t) = -\frac{5}{2} c_1 e^{-\frac{5}{2}t} \cos 6t - 6c_1 e^{-\frac{5}{2}t} \sin 6t$
$- \frac{5}{2} c_2 e^{-\frac{5}{2}t} \sin 6t + 6c_2 e^{-\frac{5}{2}t} \cos 6t$

$x'(0) = 16 \Rightarrow -\frac{5}{2} c_1 + 6c_2 = 16$

$\Rightarrow -10 + 6c_2 = 16$

$c_2 = \frac{26}{6} = \frac{13}{3}$

$x(t) = e^{-\frac{5}{2}t}\left(4 \cos 6t + \frac{13}{3} \sin(6t)\right)$

$A = \sqrt{(4)^2 + \left(\frac{13}{3}\right)^2} = \frac{\sqrt{313}}{3}$

$c_1 > 0$ & $c_2 > 0 \Rightarrow$ 1ST QUAD

$\varphi = \tan^{-1} \frac{12}{13}$

$\Downarrow$

$x(t) = \frac{\sqrt{313}}{3} e^{-\frac{5}{2}t} \cdot \sin\left(6t + \tan^{-1} \frac{12}{13}\right)$

5. (2 points) Solve. $y'' - y' - 6y = 2\sin 3x$

$$y'' - y' - 6y = 0$$

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r = 3 \qquad r = -2$$

$$y_c(x) = c_1 e^{3x} + c_2 e^{-2x}$$

$$y'' - y' - 6y = 2\sin 3x$$

$$g(x) = 2\sin 3x$$

$$y_p(x) = A\cos 3x + B\sin 3x$$

$$y_p'(x) = -3A\sin 3x + 3B\cos 3x$$

$$y_p''(x) = -9A\cos 3x - 9B\sin 3x$$

Plug in to get

$$(-9A - 3B - 6A)\cos 3x + (-9B + 3A - 6B)\sin 3x$$

$$= 2\sin 3x$$

$$-15A - 3B = 0$$

$$5(3A - 15B = 2)$$

$$-78B = 10$$

$$B = -\frac{10}{78} = \frac{-5}{39}$$

$$-15A + \frac{5}{13} = 0$$

$$A = \frac{1}{39}$$

$$y_p(x) = \frac{1}{39}\cos 3x - \frac{5}{39}\sin 3x$$

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} + \frac{1}{39}\cos 3x - \frac{5}{39}\sin 3x$$