Math 240 - Quiz 6

October 27, 2020

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This quiz is due November 5.

1. (4 points) Find the first eight nonzero terms of the power series solution centered at $x = 0$.

$$(x-2)y' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \underbrace{\sum_{n=1}^{\infty} 2n a_n x^{n-1}}_{\text{REPLACE } n \text{ BY } n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$-2a_1 + a_0 + \sum_{n=1}^{\infty} \left[(n+1)a_n - 2(n+1)a_{n+1}\right] x^n = 0$$

a_0 = ARBITRARY CONST.

$$-2a_1 + a_0 = 0$$

$$(n+1)a_n - 2(n+1)a_{n+1} = 0$$

$$\Downarrow$$

$$a_1 = \frac{1}{2} a_0$$

$$a_{n+1} = \frac{1}{2} a_n, \quad n = 1, 2, \dots$$

$$\Downarrow$$

$$a_1 = \frac{1}{2} a_0$$

$$a_2 = \frac{1}{4} a_0$$

$$a_3 = \frac{1}{8} a_0$$

$$\vdots$$

$$a_n = \frac{1}{2^n} a_0$$

$$y(x) = a_0\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + \frac{1}{64}x^6 + \frac{1}{128}x^7 + \dots\right)$$

2. (1 point) Based on the pattern in your solution above, write the power series solution as an infinite sum in summation notation.

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \frac{a_0}{1 - \frac{x}{2}}$$

Turn over.

3. (5 points) In class, we determined the power series solution of $y'' + y = 0$. The nonhomogeneous equation $y'' + y = x$ can be solved in a very similar way. Use power series to solve $y'' + y = x$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + y = x$$

$$\Downarrow$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{REPLACE } n \text{ BY } n+2} + \sum_{n=0}^{\infty} a_n x^n = x$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = x$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = x$$

$n=0 \Rightarrow 2a_2 + a_0 = 0$

$n=1 \Rightarrow 6a_3 + a_1 = 1$

$n \geq 2 \Rightarrow \underbrace{(n+2)(n+1)a_{n+2} + a_n = 0}$

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n$$

a_0 AND a_1 ARE ARBITRARY CONSTANTS.

$a_2 = -\frac{1}{2} a_0$ $\qquad$ $a_3 = -\frac{a_1 - 1}{6}$

$a_4 = \frac{1}{4 \cdot 3 \cdot 2} a_0$ $\qquad$ $a_5 = \frac{a_1 - 1}{5 \cdot 4 \cdot 3 \cdot 2}$

$a_6 = \frac{-1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_0$ $\qquad$ $a_7 = -\frac{(a_1 - 1)}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$

$\vdots$ $\qquad$ $\vdots$

$a_{2n} = \frac{(-1)^n}{(2n)!} a_0$, $n = 1, 2, 3, \ldots$ $\qquad$ $a_{2n+1} = \frac{(-1)^n (a_1 - 1)}{(2n+1)!}$, $n = 1, 2, \ldots$

$$\boxed{y(x) = a_0 + a_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 x + \sum_{n=1}^{\infty} \frac{(-1)^n (a_1 - 1)}{(2n+1)!} x^{2n+1}}$$

$$y(x) = a_0 \cos x + a_1 \sin x - \sin x + x$$