Math 240 - Quiz 8

November 12, 2020

Name key

Score ________

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. You must work individually on this quiz. This quiz is due November 17.

1. (3 points) In the forced, undamped, mass-spring system described by

$$mx'' + kx = F_0 \cos \gamma t,$$

pure resonance occurs when the natural frequency of the system, $\omega = \sqrt{k/m}$, is equal to the frequency of the external force, γ. For example, the following model describes a system in pure resonance:

$$5x'' + 320x = 80 \cos 8t; \quad x(0) = x'(0) = 0.$$

Solve the initial value problem to find the equation of motion. Use technology to graph your solution.

SEE ATTACHED SHEET.

2. (2 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} 1, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

$$F(s) = \int_0^3 e^{-st}\,dt + \int_3^\infty e^{-st}(2)\,dt = \left(-\frac{1}{s}e^{-st}\right)\Bigg|_{t=0}^{t=3} + \left(-\frac{2}{s}e^{-st}\right)\Bigg|_{t=3}^{t\to\infty}$$

$$= \frac{1}{s}e^{-st}\Bigg|_{t=3}^{t=0} + \frac{2}{s}e^{-st}\Bigg|_{t\to\infty}^{t=3}$$

$$= \frac{1}{s} - \frac{1}{s}e^{-3s} + \frac{2}{s}e^{-3s} - 0 = \frac{1}{s}\left(e^{-3s}+1\right), \quad s > 0$$

Turn over.

3. (2 points) Find the inverse Laplace transform of $F(s)$. (You may use technology to compute the partial fraction decomposition.)

$$F(s) = \frac{s+1}{s^2(s+2)^3}$$

$$\frac{1}{8}\,\frac{2}{(s+2)^{2+1}}$$

$$\frac{s+1}{s^2(s+2)^3} = \frac{1}{16}\,\frac{1}{s+2} - \frac{1}{16}\,\frac{1}{s} + \frac{1}{8}\,\frac{1}{s^2} - \boxed{\frac{1}{4}\,\frac{1}{(s+2)^3}}$$

$$f(t) = \frac{1}{16}e^{-2t} - \frac{1}{16} + \frac{1}{8}t - \frac{1}{8}t^2e^{-2t}$$

#11 #1 #2 #17

4. (3 points) Use Laplace transform techniques to solve the initial value problem.

$$y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2,\ y'(0) = 6$$

Let $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$s^2Y(s) - sy(0) - y'(0) - 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$s^2Y(s) - 2s - 6 - 6sY(s) + 12 + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = \frac{2}{(s-3)^2} + 2s - 6$$

$$Y(s) = \frac{\frac{2}{(s-3)^3} + 2s - 6}{(s-3)^2} = \frac{2}{(s-3)^5} + \frac{2}{s-3} = \frac{2}{4!}\,\frac{4!}{(s-3)^5} + 2\,\frac{1}{s-3}$$

$$y(t) = \frac{1}{12}t^4e^{3t} + 2e^{3t}$$

$\boxed{\#1}$ $x'' + 64x = 16\cos 8t$, $x(0) = x'(0) = 0$

Homogeneous eqn ...

$x'' + 64x = 0$

$r^2 + 64 = 0$

$r = \pm 8i$

$x_c(t) = c_1 \cos 8t + c_2 \sin 8t$

Non Homogeneous eqn ...

$g(t) = 16 \cos 8t$

$x_p(t) = t\left(A\cos 8t + B\sin 8t\right)$

$x_p'(t) = \left(A\cos 8t + B\sin 8t\right) + t\left(-8A\sin 8t + 8B\cos 8t\right)$

$x_p''(t) = \left(-8A\sin 8t + 8B\cos 8t\right) + \left(-8A\sin 8t + 8B\cos 8t\right) + t\left(-64A\cos 8t - 64B\sin 8t\right)$

$x_p''(t) + 64x_p(t) = 16\cos 8t$

$\Downarrow$

$16B = 16$, $-16A = 0$

$B = 1$, $A = 0$

$x_p(t) = t\sin 8t$

$x(t) = c_1\cos 8t + c_2 \sin 8t + t\sin 8t$

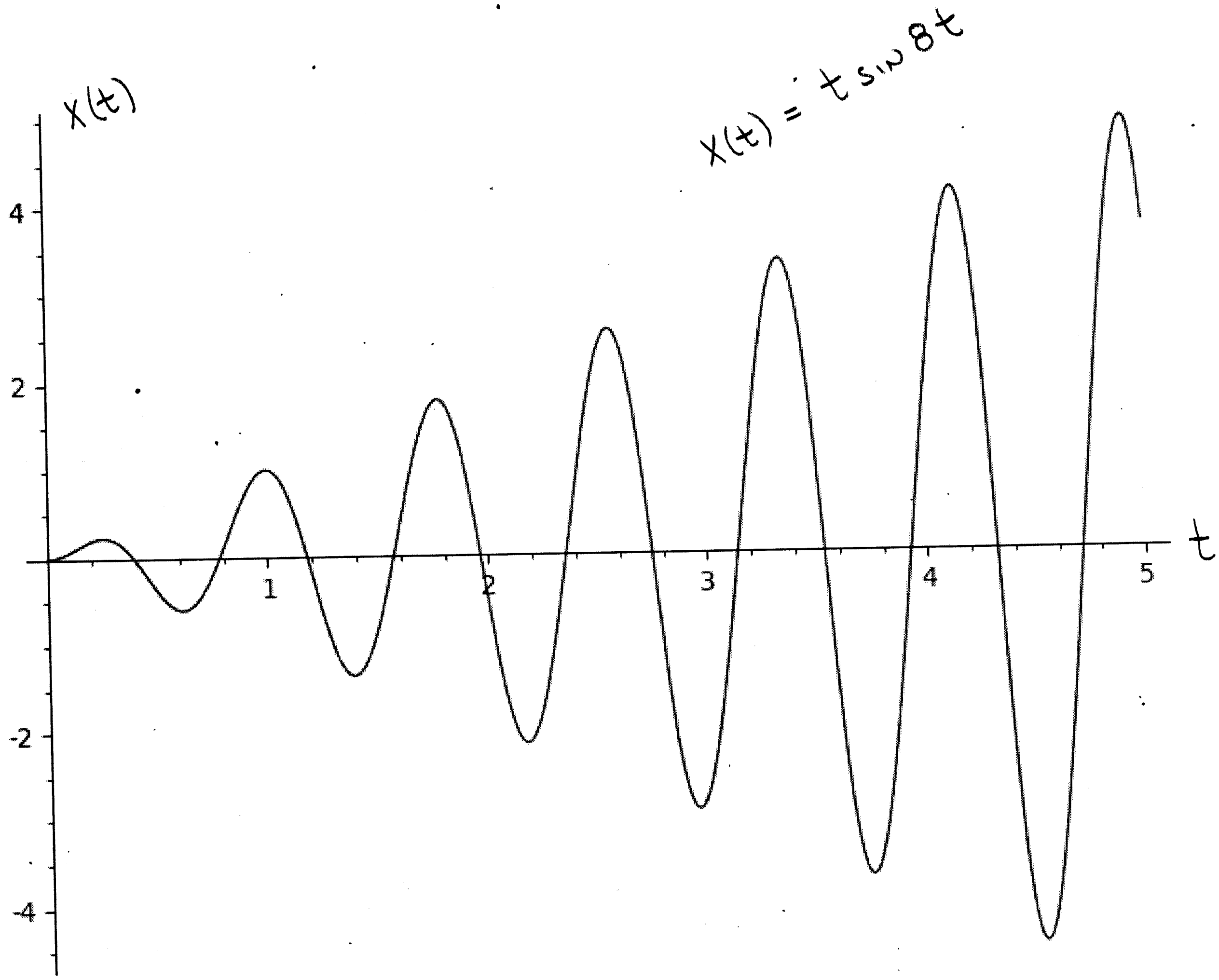
$x(0) = 0 \Rightarrow c_1 = 0$

$x'(t) = 8c_2\cos 8t + \sin 8t + 8t\cos 8t$

$x'(0) = 0 \Rightarrow c_2 = 0$

$\boxed{x(t) = t\sin 8t}$

Graph is attached.



X(t)
X(t) = t sin 8t
4
2
-2
-4
1
2
3
4
5
t