Math 240 - Quiz 9

December 3, 2020

Name key

Score

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. You must work individually on this quiz. This quiz is due by email on December 8.

1. (3 points) Use the convolution theorem to find the inverse Laplace transform of $F(s) = \dfrac{s^2}{(s^2+4)^2}$.

$$F(s) = \frac{s}{s^2+4} \cdot \frac{s}{s^2+4} \Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2\tau \cos(2t - 2\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t \left[\cos(4\tau - 2t) + \cos 2t\right] d\tau$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin(4\tau - 2t) + \tau\cos 2t\right]_{\tau=0}^{\tau=t}$$

$$\frac{1}{8}\sin 2t + \frac{1}{2}t\cos t + \frac{1}{8}\sin 2t$$

$$= \frac{1}{4}\sin 2t + \frac{1}{2}t\cos t$$

2. (2 points) Find the inverse Laplace transform of $F(s) = \ln\left(\dfrac{s-2}{s+2}\right)$.

$$F(s) = \ln(s-2) - \ln(s+2) \Rightarrow F'(s) = \frac{1}{s-2} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s+2}\right\}$$

$$= -\frac{1}{t}\left(e^{2t} - e^{-2t}\right)$$

$$= \frac{e^{-2t} - e^{2t}}{t}.$$

Turn over.

3. (5 points) Use Laplace transform techniques to solve.

LET $\mathcal{L}\{x\} = X(s)$.

$$tx'' + (t-2)x' + x = 0, \qquad x(0) = 0$$

$$\mathcal{L}\{tx''\} + \mathcal{L}\{tx'\} - 2\mathcal{L}\{x'\} + X(s) = 0$$

$$-\frac{d}{ds}\left(s^2X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{\text{CONST.}}\right) - \frac{d}{ds}\left(sX(s) - \cancel{x(0)}^{0}\right)$$

$$-2\left(sX(s) - \cancel{x(0)}^{0}\right) + X(s) = 0$$

$$-2sX(s) - s^2X'(s) - X(s) - sX'(s) - 2sX(s) + X(s) = 0$$

$$(s^2+s)X'(s) = -4sX(s) \quad \Rightarrow \quad \frac{dX}{X} = \frac{-4s}{s^2+s}\,ds = -\frac{4}{s+1}\,ds$$

$$\ln|X| = -4\ln|s+1| + C_1$$

$$\ln|X| = \ln\frac{1}{(s+1)^4} + C_1$$

$$X = \frac{C_2}{(s+1)^4}, \quad C_2 = \pm e^{C_1} \neq 0$$

$$X(s) = \frac{C_2}{(s+1)^4}$$

$$\Rightarrow \boxed{x(t) = C_3t^3e^{-t}}$$