Math 240 - Test 1
September 17, 2020

Name Key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $y(xy'' + y')(1 + x^4)^{1/2} = x$ ORDINARY, 2ND ORDER, NONLINEAR

   (b) $2e^x y - y\dfrac{dy}{dx} = 1 - x^2$ ORDINARY, 1ST ORDER, NONLINEAR

   (c) $\dfrac{\partial u}{\partial t} = \dfrac{4}{5}\dfrac{\partial^2 u}{\partial x^2}$ PARTIAL, 2ND ORDER, LINEAR

   (d) $t^2 x'' - 4tx' + 3x = t^2 \cos 6t$ ORDINARY, 2ND ORDER, LINEAR

2. (12 points) Solve the following initial value problem:

$$y\frac{dy}{dx} = 4x(y^2 + 1), \quad y(0) = 1$$

SEPARABLE...

$$\frac{y}{y^2+1}\,dy = 4x\,dx$$

$$\int \frac{y}{y^2+1}\,dy = \int 4x\,dx$$

$u = y^2+1$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|$$

$$\frac{1}{2}\ln(y^2+1) = 2x^2 + C$$

$$y^2+1 = e^{4x^2+C} = Ce^{4x^2}$$

$$y^2 = Ce^{4x^2} - 1$$

$$y(x) = \sqrt{Ce^{4x^2}-1}, \quad y>0$$

$y(0) = 1 \Rightarrow C = 2$

$$y(x) = \sqrt{2e^{4x^2}-1}$$

3. (12 points) Solve the following initial value problem:

LINEAR, 1^{ST} ORDER...

$$y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$$

$$y' - 2xy = 3x^2 e^{x^2}$$

$$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$\mu(x)\,y(x) = \int \mu(x)\,q(x)\,dx$$

$$\Downarrow$$

$$e^{-x^2} y = \int 3x^2\,dx$$

$$e^{-x^2} y = x^3 + C$$

$$y = x^3 e^{x^2} + Ce^{x^2}$$

$$y(0) = 5. \Rightarrow C = 5$$

$$y(x) = (x^3 + 5)\,e^{x^2}$$

4. (8 points) Analyze each initial value problem and determine whether we should expect a unique solution to exist through the given point.

(a) $(1-y)\dfrac{dy}{dx} = xy + 1, \quad y(2) = 1$

$$\frac{dy}{dx} = \frac{xy+1}{1-y}$$

$f(x,y) = \dfrac{xy+1}{1-y}$ NOT CONTINUOUS (OR EVEN DEFINED!) WHEN $y=1$.

A SOLUTION IS NOT GUARANTEED.

(b) $\dfrac{dy}{dx} - e^x y = x\cos^2 x, \quad y(0) = 0$

THIS IS A 1^{ST} ORDER LINEAR EQUATION

WITH $p(x) = -e^x$

AND $q(x) = x^2 \cos x$.

SINCE $p(x)$ & $q(x)$ ARE CONTINUOUS EVERYWHERE, THERE IS A UNIQUE SOLUTION FOR ANY INITIAL CONDITION.

5. (14 points) Consider the following initial value problem:

$$\overbrace{\boxed{(y^3 - y^2\sin x - x)}}^{M(x,y)}\,dx + \overbrace{\boxed{(3xy^2 + 2y\cos x)}}^{N(x,y)}\,dy = 0, \quad y(0) = 2.$$

(a) Use the test for exactness to show that the equation is exact.

$$\frac{\partial M}{\partial y} = 3y^2 - 2y\sin x \quad = \quad \frac{\partial N}{\partial x} = 3y^2 - 2y\sin x$$

EQUAL! EQUATION IS EXACT.

(b) Solve the initial value problem.

$$\frac{\partial f}{\partial x} = y^3 - y^2\sin x - x \Rightarrow f(x,y) = xy^3 + y^2\cos x - \frac{1}{2}x^2 + g(y)$$

$$\frac{\partial f}{\partial x} = 3xy^2 + 2y\cos x \Rightarrow f(x,y) = xy^3 + y^2\cos x + h(x)$$

$$h(x) = -\frac{1}{2}x^2, \quad g(y) = 0$$

$$f(x,y) = xy^3 + y^2\cos x - \frac{1}{2}x^2 = C$$

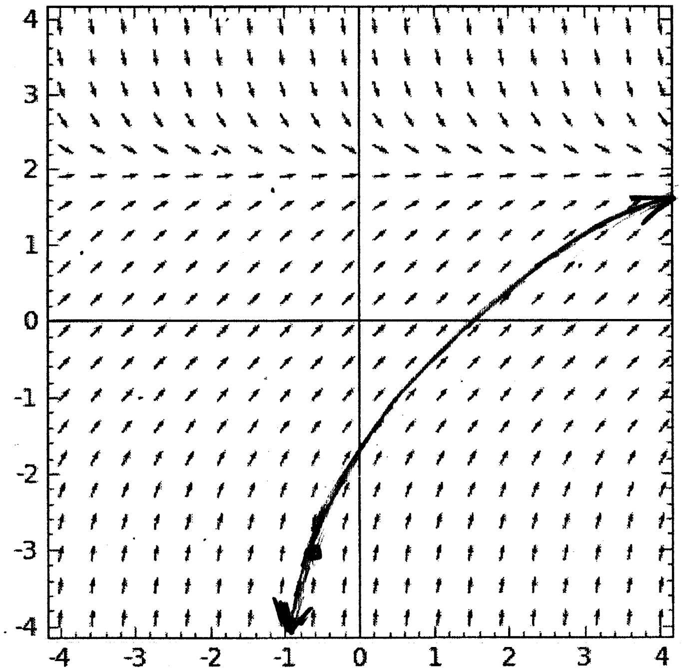
$$y(0) = 2 \Rightarrow C = 0 + 4 - 0 \Rightarrow C = 4$$

$$\boxed{xy^3 + y^2\cos x - \frac{1}{2}x^2 = 4}$$

(c) Is your solution implicit or explicit?

IT IS IMPLICIT.

6. (9 points) A direction field for $\frac{dy}{dx} = 1 - \frac{y^3}{8}$ is shown below. Use the direction field to solve the following problems.



(a) What is the unique solution passing through $(-2, 2)$?

IT LOOKS LIKE $\boxed{y(x) = 2}$ (CONSTANT FUNC.),

AND IT CHECKS OUT IN $\frac{dy}{dx} = 1 - \frac{y^3}{8}$.

(b) Suppose you have found a solution, $y(x)$, passing through a given point. What can you say about $\lim_{x\to\infty} y(x)$?

IT LOOKS LIKE $\boxed{\lim_{x\to\infty} y(x) = 2}$ REGARDLESS OF THE INITIAL CONDITION.

(c) Sketch the solution curve passing through $(-1, -3)$.

SEE ABOVE.

7. (5 points) Suppose you are sketching the direction field for the differential equation

$$\frac{dy}{dx} = \frac{5x + 10y}{-4x + 3y}.$$

(a) What is the slope of the solution curve passing through $(1, 2)$?

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{5(1) + 10(2)}{-4(1) + 3(2)} = \boxed{\frac{25}{2}}$$

(b) Give an example of a point through which a unique solution curve is not guaranteed. Briefly explain or show how you got it.

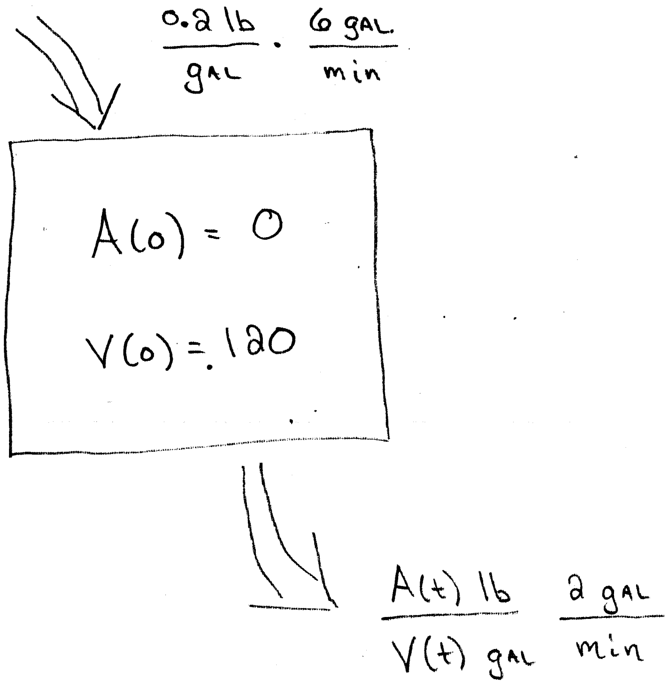
$$-4x + 3y = 0$$

HOW ABOUT $x = \frac{3}{4}$, $y = 1$ ---- $\boxed{\left(\frac{3}{4}, 1\right)}$

BECAUSE $\frac{5x + 10y}{-4x + 3y}$ IS NOT DEFINED AT $\left(\frac{3}{4}, 1\right)$, A SOLUTION IS NOT GUARANTEED.

The following problems make up the take-home portion of the test. These problems are due September 22, 2020. You must work on your own.

8. (10 points) A large tank initially contains 120 gal of pure water. A salt solution containing 0.2 lb of salt per gallon enters the tank at 6 gal/min and is uniformly mixed. The mixed solution leaves the tank at 2 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. How much salt is in the tank after 25 minutes?



$$V(t) = 120 + 4t$$

$$\frac{dA}{dt} = \text{RATE IN} - \text{RATE OUT}$$

$$\frac{dA}{dt} = 1.2 - \frac{2A}{120+4t}$$

$$\frac{dA}{dt} = 1.2 - \frac{A}{60+2t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{60+2t} A = 1.2$$

$$\mu(t) = e^{\int \frac{1}{60+2t} dt} = e^{\frac{1}{2} \ln(30+t)}$$

$$= \sqrt{30+t}$$

$$\left(\sqrt{30+t}\right) A = \int 1.2 \sqrt{30+t}\, dt$$

$$= \frac{2.4}{3} (30+t)^{3/2} + C$$

$$A = \frac{2.4}{3}(30+t) + C(30+t)^{-1/2}$$

$$A(0) = 0 \Rightarrow 24 + \frac{C}{\sqrt{30}} = 0$$

$$\Rightarrow C = -24\sqrt{30}$$

$$A(t) = \frac{2.4}{3}(30+t) - 24\sqrt{30}(30+t)^{-1/2}$$

$$A(25) \approx 26.3 \text{ lb}$$

9. (10 points) Solve: $xy^2y' + y^3 = x\cos x$
(If/when appropriate, you may assume $x > 0$.)

$$y^2 y' + \frac{1}{x} y^3 = \cos x$$

BERNOULLI WITH $n = -2$

$$\left.\begin{array}{l} \text{LET } u = y^3 \\ \dfrac{du}{dx} = 3y^2 y' \end{array}\right\} \quad \frac{1}{3}u' + \frac{1}{x}u = \cos x$$

$$u' + \frac{3}{x}u = 3\cos x$$

$$\mu(x) = e^{\int \frac{3}{x}dx}$$
$$= e^{3\ln|x|} = |x|^3$$
$$= x^3, \; x > 0$$

$$x^3 u = \int 3x^3 \cos x \, dx$$

TABULAR PARTS

SIGNS	u & DERIVS	dv/dx AND ANTID'S
$+$	$3x^3$	$\cos x$
$-$	$9x^2$	$\sin x$
$+$	$18x$	$-\cos x$
$-$	18	$-\sin x$
$+$	0	$\cos x$

$$\int 3x^3 \cos x \, dx = 3x^3 \sin x + 9x^2 \cos x - 18x \sin x - 18\cos x + C$$

$$u = 3\sin x + \frac{9}{x}\cos x - \frac{18}{x^2}\sin x - \frac{18}{x^3}\cos x + \frac{C}{x^3}$$

$u = y^3$

$$\boxed{y(x) = \sqrt[3]{3\sin x + \frac{9}{x}\cos x - \frac{18}{x^2}\sin x - \frac{18}{x^3}\cos x + \frac{C}{x^3}}}$$

10. (10 points) Solve: $xy' = y + 2xe^{-y/x}$

$$y' = \frac{y}{x} + 2e^{-y/x}$$

HOMOGENEOUS.

LET $u = \frac{y}{x}$

$$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + xu' = u + 2e^{-u}$$

$$xu' = 2e^{-u} \qquad \text{(SEPARABLE)}$$

$$e^{u}\,du = \frac{2}{x}\,dx$$

$$\int e^{u}\,du = \int \frac{2}{x}\,dx$$

$$e^{u} = 2\ln|x| + C$$

$$e^{u} = \ln x^2 + C$$

$$u = \ln(\ln x^2 + C)$$

$u = y/x$

$$\Rightarrow \boxed{y(x) = x\ln(\ln x^2 + C)}$$