Math 240 - Test 2

October 15, 2020

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Consider the equation $(x^2+1)y'' - 2xy' + 2y = 0$.

   (a) Verify that $y_1(x) = x$ and $y_2(x) = 1 - x^2$ are solutions.

$y(x) = x \ldots$

$y'(x) = 1$

$y''(x) = 0$

$(x^2+1)(0) - 2x(1) + 2x$

$= -2x + 2x = 0$ ✓

$y(x) = 1 - x^2$

$y'(x) = -2x$

$y''(x) = -2$

$(x^2+1)(-2) - 2x(-2x) + 2(1-x^2)$

$= -2x^2 - 2 + 4x^2 + 2 - 2x^2$

$= 0$ ✓

   (b) Use the Wronskian to show that y_1 and y_2 are linearly independent.

$$W = \begin{vmatrix} x & 1-x^2 \\ 1 & -2x \end{vmatrix} = -2x^2 - (1-x^2)$$

$$= -x^2 - 1 \leq -1 \neq 0 \Rightarrow$$ y_1 & y_2 ARE LIN. INDEP.

   (c) Use what you've learned in parts (a) and (b) to find the solution of the IVP $(x^2+1)y'' - 2xy' + 2y = 0;\ y(1) = 1,\ y'(1) = -1$.

$y(x) = c_1 x + c_2(1-x^2)$

$y(1) = 1 \Rightarrow c_1 = 1$

$y'(x) = 1 + c_2(-2x)$

$y'(1) = -1 \Rightarrow 1 - 2c_2 = -1$

$c_2 = 1$

$\boxed{y(x) = x + 1 - x^2}$

   (d) Is your solution in part (c) unique? Explain.

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = 0$$

YES! P(x) AND Q(x) ARE CONTINUOUS EVERYWHERE. By OUR EXISTENCE/UNIQUENESS THEOREM FOR LINEAR EQUATIONS, THERE IS A UNIQUE SOLUTION DEFINED FOR ALL REAL #'S.

2. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $x(t) = 2e^{-2t} + 5te^{-2t}$ CRITICALLY DAMPED

THE CHAR. EQ'N HAS A REPEATED SOLUTION. DISC. = 0

(b) $x'' + 8x' + 17x = 0$

$\text{Disc} = 8^2 - 4(1)(17) = -4 < 0$ UNDERDAMPED

(c) $x(t) = \sqrt{6}\sin(4t + \pi)$

No DAMPING SIMPLE HARMONIC MOTION

(d) $2x'' + 5x' + 3x = 0$

$\text{Disc} = 5^2 - 4(2)(3) = 1 > 0$ OVERDAMPED

3. (8 points) Find the general solution: $y^{(4)} - 2y''' + y'' = 0$

CHAR EQ'N IS $r^4 - 2r^3 + r^2 = 0$

$r^2(r^2 - 2r + 1) = 0$

$r^2(r-1)^2 = 0 \qquad r = 0, 0, 1, 1$

$$y(x) = C_1 + C_2 x + C_3 e^x + C_4 x e^x$$

4. (8 points) Solve the following initial value problem.

$$2y'' - 2y' + y = 0; \quad y(0) = -1,\, y'(0) = 0$$

CHAR EQN IS

$$2r^2 - 2r + 1 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(2)(1)}}{4}$$

$$= \frac{2 \pm \sqrt{-4}}{4} = \frac{1}{2} \pm \frac{1}{2}i$$

$$y(x) = c_1 e^{x/2} \cos \tfrac{1}{2}x + c_2 e^{x/2} \sin \tfrac{1}{2}x$$

$$y(0) = -1 \Rightarrow c_1 = -1$$

$$y'(x) = -\tfrac{1}{2} e^{x/2} \cos \tfrac{1}{2}x + \tfrac{1}{2} e^{x/2} \sin \tfrac{1}{2}x + \tfrac{1}{2} c_2 e^{x/2} \sin \tfrac{1}{2}x + \tfrac{1}{2} c_2 e^{x/2} \cos \tfrac{1}{2}x$$

$$y'(0) = 0 \Rightarrow -\tfrac{1}{2} + \tfrac{1}{2} c_2 = 0$$

$$c_2 = 1$$

$$\boxed{y(x) = e^{x/2}\left(\sin \tfrac{1}{2}x - \cos \tfrac{1}{2}x\right)}$$

5. (5 points) For $x > 0$, let $y_1(x) = \ln x^5$ and $y_2(x) = \ln x$. Compute the Wronskian of y_1 and y_2. Briefly explain why $y(x) = c_1 y_1(x) + c_2 y_2(x)$ cannot be the general solution of a 2nd-order, linear, homogeneous differential equation.

$$W = \begin{vmatrix} \ln x^5 & \ln x \\ \frac{5x^4}{x^5} & \frac{1}{x} \end{vmatrix} = \frac{\ln x^5}{x} - \frac{5x^4 \ln x}{x^5}$$

$$= \frac{5 \ln x}{x} - \frac{5 \ln x}{x} = 0$$

y_1 AND y_2 ARE NOT LINEARLY INDEPENDENT.

IN ORDER TO HAVE THE GENERAL SOLUTION OF A 2ND ORDER, LINEAR, HOMOGENEOUS EQUATION, WE NEED A LINEAR COMBINATION OF TWO LINEARLY INDEPENDENT SOLUTIONS.

6. (12 points) Find the general solution: $y'' - 5y' + 4y = 2e^{4x}$

CORRESPONDING HOMOGENEOUS EQUATION: $y'' - 5y' + 4y = 0$

CHAR. EQN: $r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 4, \ r = 1$

$y_c(x) = c_1 e^{4x} + c_2 e^{x}$

NONHOMOGENEOUS EQN:

RHS: $g(x) = 2e^{4x}$

$y_p(x) = x^s A e^{4x}$

MUST CHOOSE $s = 1$

$y_p(x) = Axe^{4x}$

$y_p'(x) = Ae^{4x} + 4Axe^{4x}$
$= Ae^{4x} + 4y_p(x)$

$y_p''(x) = 4Ae^{4x} + 4y_p'(x)$
$= 8Ae^{4x} + 16y_p(x)$

$y_p'' - 5y_p' + 4y = 2e^{4x}$

$\Downarrow$

$[8Ae^{4x} + 16y_p(x)] - 5[Ae^{4x} + 4y_p(x)] + 4y_p(x)$

$= (8A - 5A)e^{4x} = 2e^{4x}$

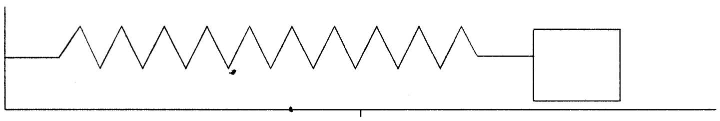
$\Downarrow$

$3A = 2 \quad A = \frac{2}{3}$

$y_p(x) = \frac{2}{3} x e^{4x}$

$y(x) = c_1 e^{4x} + c_2 e^{x} + \frac{2}{3} x e^{4x}$

7. *(12 points) A 1-kg mass is attached to a spring with spring constant* 16 N/m. The damping constant for the system is 10 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and pushed to the left at 12 m/sec. Find the equation of motion. Is the system underdamped, overdamped, or critically damped? How do you know?



$$1x'' + 10x' + 16x = 0, \quad x(0) = 1, \quad x'(0) = -12$$

CHAR EQ'N IS $r^2 + 10r + 16 = 0$

$$(r+2)(r+8) = 0$$

$r = -2,\ r = -8$ $\longrightarrow$ TWO UNIQUE REAL SOLUTIONS

$\Longrightarrow$ $\boxed{\text{SYSTEM IS OVERDAMPED}}$

$$x(t) = c_1 e^{-2t} + c_2 e^{-8t}$$

$$x(0) = 1 \Rightarrow c_1 + c_2 = 1$$

$$x'(t) = -2c_1 e^{-2t} - 8c_2 e^{-8t}$$

$$x'(0) = -12 \Rightarrow -2c_1 - 8c_2 = -12$$

$$x(t) = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^{-8t}$$

$$2(c_1 + c_2 = 1)$$
$$-2c_1 - 8c_2 = -12$$
$$-6c_2 = -10$$
$$c_2 = \frac{5}{3}$$

$$c_1 + c_2 = 1$$
$$c_1 = -\frac{2}{3}$$

The following problems make up the take-home portion of the test. These problems are due October 20, 2020. You must work on your own.

8. (5 points) Consider the following equation:

$$y'' + 4y = x \cos x + \cos 2x.$$

Solve the corresponding homogeneous equation. Then find the appropriate <u>form</u> of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

Corresponding Homogeneous Eqn:

$$y'' + 4y = 0$$

Char. Eqn: $r^2 + 4 = 0$

$$r = \pm 2i$$

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$$

NonHomogeneous Eqn:

① $g(x) = x \cos x$

$$\Downarrow$$

$$y_{p_1}(x) = (Ax + B)\cos x + (Cx + D)\sin x$$

② $g(x) = \cos 2x$

$$\Downarrow$$

$$y_{p_2}(x) = x\left(E \cos 2x + F \sin 2x\right)$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + (Ax + B)\cos x + (Cx + D)\sin x + Ex \cos 2x + Fx \sin 2x$$

9. (10 points) Use variation of parameters to solve the following differential equation.

$$x'' - 4x' + 4x = \frac{e^{2t}}{t^2}, \quad t > 0$$

CORRESPONDING HOMOGENEOUS EQUATION:

$$x'' - 4x' + 4x = 0$$

CHAR EQN:

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2, 2$$

$$x_c(t) = c_1 e^{2t} + c_2 t e^{2t}$$

NONHOMOGENEOUS EQN HAS

$$g(t) = \frac{e^{2t}}{t^2}$$

VARIATION OF PARAMETERS FORMULA ...

$$W(x_1, x_2) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix}$$

$$= e^{4t}$$

$$V_1(t) = \int \frac{-\frac{e^{2t}}{t^2}(te^{2t})}{e^{4t}}\, dt$$

$$= \int -\frac{1}{t}\, dt = -\ln t$$

$$V_2(t) = \int \frac{\frac{e^{2t}}{t^2}(e^{2t})}{e^{4t}}\, dt$$

$$= \int \frac{1}{t^2}\, dt = -\frac{1}{t}$$

$$x_p(t) = -e^{2t}\ln t - te^{2t}\left(\frac{1}{t}\right)$$

$$= -e^{2t}\ln t - e^{2t}$$

GENERAL SOLUTION IS

$$\boxed{x(t) = c_3 e^{2t} + c_2 t e^{2t} - e^{2t}\ln t}$$

10. (10 points) Solve: $y'' - 2y' + 5y = 5x + 8 + x \sin x$

$y'' - 2y' + 5y = 0$

$r^2 - 2r + 5 = 0$

$r^2 - 2r + 1 = -4$

$(r-1)^2 = -4$

$r = 1 \pm 2i$

$y_1(x) = e^x \cos 2x$

$y_2(x) = e^x \sin 2x$

$y_c(x) = e^x (c_1 \cos 2x + c_2 \sin 2x)$

$g(x) = 5x + 8$

$\Downarrow$

$y_{p_1}(x) = Ax + B$

$y_{p_1}'' - 2y_{p_1}' + 5y_{p_1}$

$= 0 - 2A + 5(Ax + B)$

$= 5x + 8$

$5A = 5$

$-2A + 5B = 8$

$A = 1,\ B = 2$

$y_{p_1}(x) = x + 2$

$g(x) = x \sin x$

$\Downarrow$

$y_{p_2}(x) = (Ax + B)\cos x + (Cx + D)\sin x$

$y_{p_2}'(x) = (A + Cx + D)\cos x + (-Ax - B + C)\sin x$

$y_{p_2}''(x) = (-Ax - B + 2C)\cos x + (-2A - Cx - D)\sin x$

$y_{p_2}'' - 2y_{p_2}' + 5y_{p_2} = x \sin x$

$\Downarrow$

$(-Ax - B + 2C) - 2(A + Cx + D) + 5(Ax + B) = 0$

&

$(-2A - Cx - D) - 2(-Ax - B + C) + 5(Cx + D) = x$

OVER $\longrightarrow$

CONTINUING UP THERE

$$4A - 2C = 0 \Rightarrow \quad C = 2A$$

$$-2A + 4B + 2C - 2D = 0$$

$$2A + 4C = 1 \qquad 2A + 8A = 1 \Rightarrow A = \frac{1}{10}$$

$$C = \frac{1}{5}$$

$$-2A + 2B - 2C + 4D = 0$$

ADD AND SUBTRACT ROWS 2 & 4 TO GET

$$2B + 4C - 6D = 0$$

$$-4A + 6B + 2D = 0$$

$$2B - 6D = -\frac{4}{5}$$

$$6B + 2D = \frac{2}{5}$$

$$20B = \frac{2}{5} \qquad 20D = \frac{14}{5}$$

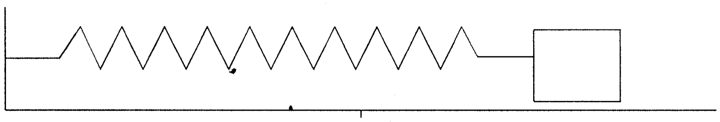
$$B = \frac{2}{100} \qquad D = \frac{14}{100}$$

$$y_{P_2}(x) = \left(\frac{1}{10}x + \frac{1}{50}\right)\cos x + \left(\frac{1}{5}x + \frac{7}{50}\right)\sin x$$

GENERAL SOLUTION IS

$$\boxed{y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x + x + 2 + \left(\frac{1}{10}x + \frac{1}{50}\right)\cos x + \left(\frac{1}{5}x + \frac{7}{50}\right)\sin x}$$

11. (10 points) A 1/2-kg mass is attached to a spring with spring constant 5 N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the left of equilibrium (compressing the spring) and released from rest. Find the equation of motion. Write your final result in terms of a single trig function with a phase shift. Use technology to graph your solution and attach a copy of the graph.



$x = 0$
(Equilibrium)

$$\tfrac{1}{2}x'' + x' + 5x = 0; \quad x(0) = -2, \; x'(0) = 0$$

$$x'' + 2x' + 10x = 0$$

$$r^2 + 2r + 10 = 0$$

$$r^2 + 2r + 1 = -9$$

$$(r+1)^2 = -9$$

$$r = -1 \pm 3i$$

$$x(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$$

$$x(0) = -2 \Rightarrow c_1 = -2$$

$$x'(t) = -c_1 e^{-t}\cos 3t - 3c_1 e^{-t}\sin 3t - c_2 e^{-t}\sin 3t + 3c_2 e^{-t}\cos 3t$$

$$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0$$
$$2 + 3c_2 = 0 \Rightarrow c_2 = -\tfrac{2}{3}$$

$$x(t) = -2e^{-t}\cos 3t - \tfrac{2}{3}e^{-t}\sin 3t$$

$$A = \sqrt{(-2)^2 + \left(-\tfrac{2}{3}\right)^2} = \sqrt{\tfrac{40}{9}} = \tfrac{2}{3}\sqrt{10}$$

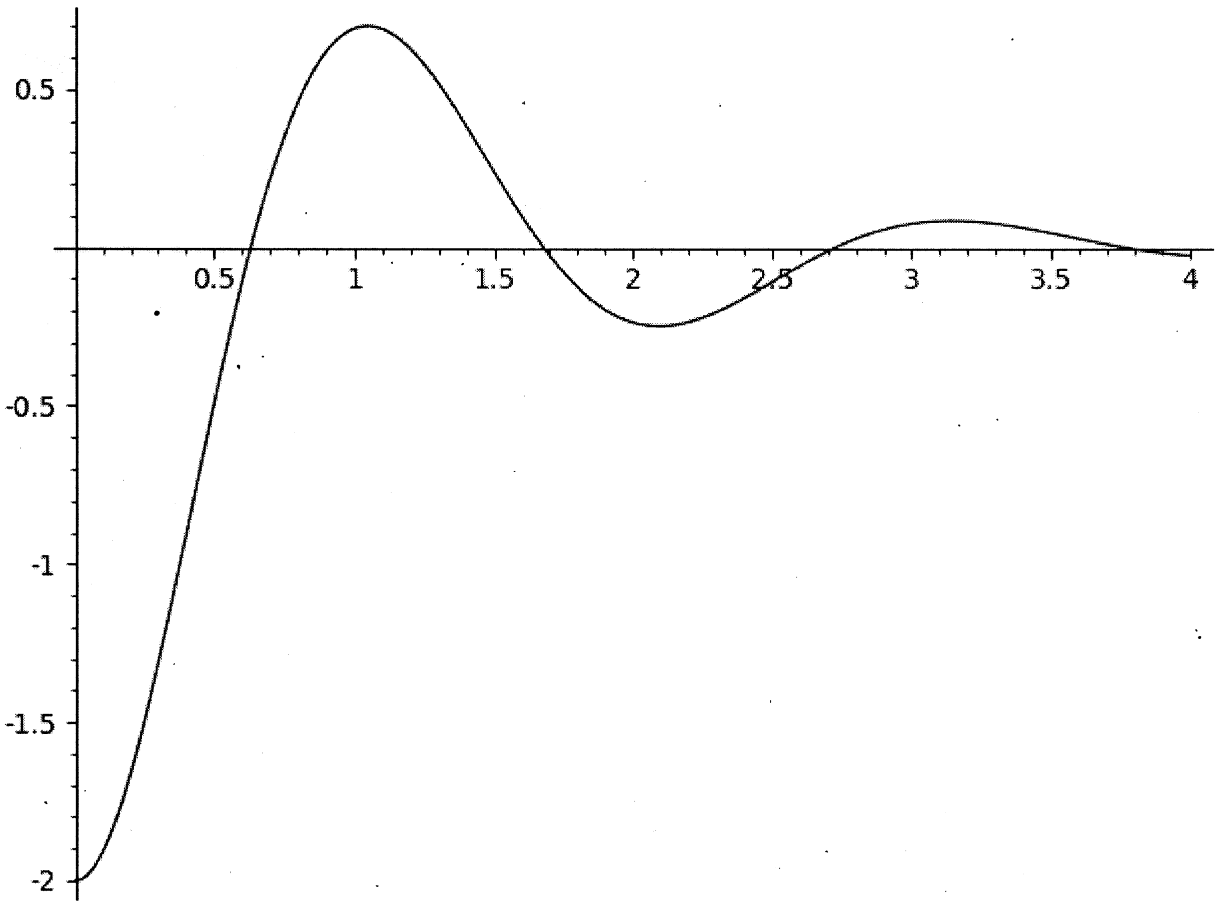
$$\tan\varphi = \frac{-2}{-2/3} = 3$$

$$c_1, c_2 < 0 \Rightarrow \varphi \text{ in Quad } 3.$$

$$\varphi = \pi + \tan^{-1} 3$$

$$x(t) = \frac{2\sqrt{10}}{3} e^{-t} \sin\left(3t + \pi + \tan^{-1} 3\right)$$

SEE ATTACHED GRAPH.


0.5
-0.5
-1
-1.5
-2
0.5
1
1.5
2
2.5
3
3.5
4