Math 240 - Test 3A

November 19, 2020

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use Laplace transform techniques to solve.

$$y'' + 4y' + 4y = t^3e^{-2t}; \quad y(0) = 5,\ y'(0) = -10$$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{t^3 e^{-2t}\}$$

$$s^2 Y(s) - 5s + 10 + 4sY(s) - 20 + 4Y(s) = \frac{6}{(s+2)^4}$$

$$(s^2 + 4s + 4)Y(s) - 5s - 10 = \frac{6}{(s+2)^4}$$

$$(s+2)^2 Y(s) = \frac{6}{(s+2)^4} + 5(s+2)$$

$$Y(s) = \frac{6}{(s+2)^6} + \frac{5}{s+2}$$

$$\Downarrow$$

$$y(t) = \frac{6}{5!} t^5 e^{-2t} + 5e^{-2t}$$

$$\boxed{y(t) = \frac{1}{20} t^5 e^{-2t} + 5e^{-2t}}$$

$$y'' + \frac{x}{x^2-2}y' - \frac{16}{x^2-2}y = 0$$

SING. PTS.

$x = \pm\sqrt{2}$

RADIUS OF CON. IS AT LEAST $\sqrt{2}$

2. (10 points) State the recurrence relation that describes the coefficients of the power series solution, and state the guaranteed (by our theorem) radius of convergence.

$$(x^2-2)y'' + xy' - 16y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x^2-2)y'' + xy' - 16y = 0$$

$$\Downarrow$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$n \to n+2$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$(-4a_2 - 16a_0) + (-12a_3 + a_1 - 16a_1)x$$

$$+ \sum_{n=2}^{\infty} \left[n(n-1)a_n - 2(n+2)(n+1)a_{n+2} + na_n - 16a_n\right] x^n = 0$$

$$-4a_2 - 16a_0 = 0 \Rightarrow a_2 = -4a_0$$

$$-12a_3 - 15a_1 = 0 \Rightarrow a_3 = -\frac{5}{4}a_1$$

$$-2(n+2)(n+1)a_{n+2} = -[\underbrace{n(n-1)+n-16}_{n^2-16}]a_n$$

a_0 & a_1 ARE ARBITRARY

$a_2 = -4a_0$

$$a_{n+2} = \frac{n^2-16}{2(n+2)(n+1)} a_n, \quad n = 2, 3, 4, \ldots$$

$a_3 = -\frac{5}{4}a_1$

Math 240 - Test 3B
November 19, 2020

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. This test is due December 1 by email. **You must work individually on this test.**

1. (15 points) Find a power series solution. Then refer to the Common Infinite Series handout to find the interval of convergence and a familiar expression for your solution.

$$(x-10)y' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xy' - 10y' + y =$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 10 n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n =$$

$n+1 \rightarrow n$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 10(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 10(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} \left[n a_n - 10(n+1) a_{n+1} + a_n \right] x^n = 0$$

$$n a_n - 10(n+1) a_{n+1} + a_n = 0$$

$$a_{n+1} = \frac{1}{10} a_n, \quad n = 0, 1, 2, 3, \ldots$$

a_0 = ARBITRARY.

$$y(x) = a_0 + a_0 \frac{x}{10} + a_0 \frac{x^2}{100} + a_0 \frac{x^3}{10^3} + \cdots$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$$

$$= \frac{a_0}{1 - \frac{x}{10}} = \frac{10 a_0}{10 - x},$$

$$-10 < x < 10$$

2. (20 points) Consider the following first-order linear equation.

$$x^3 y' = 2y$$

(a) Find a power series solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$x^3 y' - 2y = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n+2} - \sum_{n=0}^{\infty} 2 a_n x^n =$$

$n-2 \to n$

$$\sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - \sum_{n=0}^{\infty} 2 a_n x^n.$$

$$= -2a_0 - 2a_1 x - 2a_2 x^2 + \sum_{n=3}^{\infty} \left[(n-2) a_{n-2} - 2a_n\right] x^n$$

$$= 0$$

$a_0 = 0,\ a_1 = 0,\ a_2 = 0.$

$$a_n = \frac{n-2}{2} a_{n-2}, \quad n = 3, 4, 5, \ldots$$

$$\Rightarrow a_n = 0 \Rightarrow \boxed{y(x) = 0}$$

(b) Use the techniques of section 1.5 to find the general solution.

$$y' - \frac{2}{x^3} y = 0$$

$$\mu(x) = e^{\int -\frac{2}{x^3} dx} = e^{1/x^2}$$

$$e^{1/x^2} y(x) = \int 0 \, dx$$

$$e^{1/x^2} y(x) = C$$

$$\boxed{y(x) = Ce^{-1/x^2}}$$

(c) Compare your solutions. Why did the power series approach fail to provide the general solution?

X = 0 IS A SINGULAR POINT. WE CANNOT EXPECT A POWER SERIES TO GIVE A SOLUTION CENTERED AT A SINGULAR POINT.

3. (8 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} 1-t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^1 (1-t)e^{-st}\,dt + \int_1^\infty 0\, e^{-st}\,dt$$

$$= \int_0^1 (1-t)e^{-st}\,dt$$

$u = 1-t \qquad du = -dt$

$dv = e^{-st}\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$\frac{t-1}{s}e^{-st}\Big|_{t=0}^{t=1} - \int_0^1 \frac{1}{s}e^{-st}\,dt$$

$$= 0 - \left(-\frac{1}{s}\right) - \left(-\frac{1}{s^2}e^{-st}\right)\Big|_{t=0}^{t=1}$$

$$\frac{1}{s} - \left(-\frac{e^{-s}}{s^2} + \frac{1}{s^2}\right)$$

$$= \boxed{\frac{e^{-s}}{s^2} - \frac{1}{s^2} + \frac{1}{s}}$$

4. (15 points) Use Laplace transform techniques to solve the following initial value problem. You may use technology for any partial fraction decompositions, but do everything else by hand.

$$y'' + 6y' + 18y = \cos 2t; \quad x(0) = 1,\ x'(0) = -1$$ SHOULD BE y'S.

TAKE LAPLACE TRANSFORM OF BOTH SIDES:

$$s^2 Y(s) - s(1) - (-1) + 6\left[sY(s) - 1\right] + 18Y(s) = \frac{s}{s^2+4}$$

$$(s^2 + 6s + 18)Y(s) - s - 5 = \frac{s}{s^2+4}$$

$$Y(s) = \frac{\frac{s}{s^2+4} + s + 5}{s^2 + 6s + 18}$$

PFD FROM Sage ...

$$Y(s) = \frac{163s + 796}{170\left[(s+3)^2 + 9\right]} + \frac{7s + 12}{170(s^2+4)}$$

$$Y(s) = \frac{163(s+3)}{170\left[(s+3)^2+9\right]} + \frac{307}{170\left[(s+3)^2+9\right]} + \frac{7s}{170(s^2+4)} + \frac{12}{170(s^2+4)}$$

INVERSE TRANSFORM FROM TABLE ...

$$y(t) = \frac{163}{170} e^{-3t} \cos 3t + \frac{307}{170} \cdot \frac{1}{3} e^{-3t} \sin 3t + \frac{7}{170} \cos 2t + \frac{12}{170} \cdot \frac{1}{2} \sin 2t$$

$$y(t) = \frac{1}{170} e^{-3t}\left(163 \cos 3t + \frac{307}{3} \sin 3t\right) + \frac{1}{170}\left(7 \cos 2t + 6 \sin 2t\right)$$

5. (8 points) Use Laplace transform techniques to solve the following system of initial value problems.

$$x' = x + 2y, \quad x(0) = 0$$
$$y' = x + e^{-t}, \quad y(0) = 0$$

$$\mathcal{L}\{x'\} = \mathcal{L}\{x + 2y\} \Rightarrow sX(s) = X(s) + 2Y(s)$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{x + e^{-t}\} \Rightarrow sY(s) = X(s) + \frac{1}{s+1}$$

$$(s-1)X(s) - 2Y(s) = 0$$
$$-X(s) + sY(s) = \frac{1}{s+1}$$

USE ELIMINATION TO SOLVE SYSTEM ...

MULT TOP EQN BY s AND BOTTOM BY 2. THEN ADD

$$(s^2 - s - 2)X(s) = \frac{2}{s+1}$$

$$(s+1)(s-2)X(s) = \frac{2}{s+1}$$

$$X(s) = \frac{2}{(s+1)^2(s-2)} \underset{\text{PFD BY SAGE}}{=} \frac{-2/9}{s+1} - \frac{2/3}{(s+1)^2} + \frac{2/9}{s-2}$$

$$x(t) = -\frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} + \frac{2}{9}e^{2t}$$

TOP EQUATION THEN GIVES

$$Y(s) = \frac{s-1}{(s+1)^2(s-2)} \underset{\text{PFD BY SAGE}}{=} \frac{-1/9}{s+1} + \frac{2/3}{(s+1)^2} + \frac{1/9}{s-2}$$

$$y(t) = -\frac{1}{9}e^{-t} + \frac{2}{3}te^{-t} + \frac{1}{9}e^{2t}$$

6. (14 points) Let $F(s) = \dfrac{3}{s(s+5)}$.

(a) Find the inverse Laplace transform by first computing the partial fraction decomposition (by hand).

$$\frac{3}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$3 = A(s+5) + Bs$$

$$s=0 \Rightarrow \quad 3 = 5A \Rightarrow A = 3/5$$

$$s=-5 \Rightarrow \quad 3 = -5B \Rightarrow B = -3/5$$

$$F(s) = \frac{3/5}{s} - \frac{3/5}{s+5}$$

$$\boxed{f(t) = \frac{3}{5} - \frac{3}{5}e^{-5t}}$$

(b) Find the inverse Laplace transform by using Theorem 2 on page 284.

LET $G(s) = \dfrac{3}{s+5}$.

$$g(t) = 3e^{-5t}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = \int_0^t 3e^{-5\tau}\,d\tau = -\frac{3}{5}e^{-5\tau}\Big|_0^t = \boxed{-\frac{3}{5}e^{-5t} + \frac{3}{5}}$$

(c) Apply the convolution theorem to find the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{1}{s+5}\cdot\frac{3}{s}\right\} = x(t) * y(t)$$ WHERE $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$

AND $y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3$

$$f(t) = x(t) * y(t) = \int_0^t e^{-5\tau}\cdot 3\,d\tau = -\frac{3}{5}e^{-5\tau}\Big|_0^t$$

$$= \boxed{-\frac{3}{5}e^{-5t} + \frac{3}{5}}$$