Math 240 - Final Exam

December 17, 2020

Name key

Score

Show all work to receive full credit. You must work individually. This test is due no later than December 18 at 11:59 pm. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD.

---

1. (12 points) According to Newton's Law of Cooling, the temperature T at time t of an object cooling in a medium of constant temperature M is described by the differential equation
$$\frac{dT}{dt} = k(M - T),$$
where k is some constant.

(a) Solve the differential equation.

$$\frac{dT}{M-T} = k\,dt$$

$$-\ln|M-T| = kt + C$$

$$\ln|M-T| = -kt - C_1$$

$$|M-T| = e^{-C_1}e^{-kt} = C_2 e^{-kt}$$

$$M - T = C_3 e^{-kt}$$

$$T(t) = M - C_3 e^{-kt}$$

(b) An object at 120°F is moved into a large room with an ambient temperature of 72°F. The object cools to 100°F in 6 min. Use your result from part (a) to find a formula for the temperature of the object at time t.

$$T(t) = 72 - Ce^{-kt}$$

$$T(0) = 120 = 72 - C$$

$$\Rightarrow C = -48$$

$$T(t) = 72 + 48e^{-kt}$$

$$T(6) = 100 \Rightarrow 28 = 48e^{-6k}$$

$$k = \frac{\ln 28/48}{-6} = \frac{\ln 7/12}{-6}$$

$$T(t) = 72 + 48e^{\frac{\ln 7/12}{6}t}$$

(c) When will the object reach 76°F?

$$76 = 72 + 48e^{\frac{\ln\frac{7}{12}}{6}t}$$

$$\Rightarrow \frac{4}{48} = e^{\frac{\ln\frac{7}{12}}{6}t}$$

$$\Rightarrow \ln\frac{1}{12} = \frac{1}{6}\ln\frac{7}{12}\,t$$

$$t = \frac{-6\ln 12}{\ln 7 - \ln 12}$$

$$\approx 27.6 \text{ min}$$

2. (20 points) The following differential equation falls into at least two of the named types of equations in section 1.6. Use two different approaches from section 1.6 to solve the equation. Use a word or short phrase to describe each approach.

$$2xyy' = x^2 + 2y^2$$

(a) First approach:

$$y' = \frac{x^2}{2xy} + \frac{2y^2}{2xy}$$

$$y' - \frac{1}{x}y = \frac{1}{2}xy^{-1}$$

BERNOULLI

LET $u = y^2$

$u' = 2yy'$

$xu' = x^2 + 2u$

$$u' - \frac{2}{x}u = x$$

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$\frac{1}{x^2}u(x) = \int \frac{1}{x^2}x\,dx$$

$$\frac{1}{x^2}u(x) = \ln|x| + C$$

$$u(x) = x^2\ln|x| + Cx^2$$

$$y(x) = \pm\sqrt{x^2\ln|x| + Cx^2}$$

(b) Second approach:

$$y' = \frac{x^2}{2xy} + \frac{2y^2}{2xy}$$

$$y' = \frac{1}{2}\frac{x}{y} + \frac{y}{x}$$

Homogeneous

$u = \frac{y}{x} \Rightarrow ux = y$

$y' = u + xu'$

$$u + xu' = \frac{1}{2}\frac{1}{u} + u$$

$$x\frac{du}{dx} = \frac{1}{2}\frac{1}{u}$$

$$u\,du = \frac{1}{2}\frac{1}{x}$$

$$\frac{u^2}{2} = \frac{1}{2}\ln|x| + C$$

$$u^2 = \ln|x| + C$$

$$\frac{y^2}{x^2} = \ln|x| + C$$

$$y^2 = x^2\ln|x| + Cx^2$$

$$y(x) = \pm\sqrt{x^2\ln|x| + Cx^2}$$

3. (21 points) For any nonzero constant k, the equation $y' + ky = 0$ describes exponential growth or decay. This semester, we have studied at least 5 different ways to solve this equation. Solve the equation using three different approaches. Use a word or short phrase to describe each approach.

(a) First approach: $\boxed{\text{SEPARABLE}}$

$$y' = -ky \rightarrow \frac{dy}{y} = -k\,dx$$

$$\ln|y| = -kx + C$$

$$|y| = e^{C} e^{-kx}$$

$$y = C_2 e^{-kx}$$

$$\boxed{y(x) = C_2 e^{-kx}}$$

(b) Second approach: $\boxed{\text{LINEAR, 1ST-ORDER}}$

$$y' + ky = 0$$

$$\mu(x) = e^{\int k\,dx} = e^{kx}$$

$$e^{kx} y(x) = \int 0 e^{kx}\,dx$$

$$e^{kx} y(x) = C$$

$$\boxed{y(x) = Ce^{-kx}}$$

(c) Third approach: $\boxed{\text{LAPLACE TRANSFORM}}$

$$y' + ky = 0$$

$$sY(s) - y(0) + Y(s) = 0$$

$$(s+1)Y(s) = y(0)$$

$$Y(s) = \frac{y(0)}{s+1}$$

$$\Downarrow$$

$$\boxed{y(t) = y(0)e^{-kt}}$$

SOME OTHER APPROACHES...

- CONST. COEFF, LINEAR, HOMOGENEOUS --- CHAP 2
- POWER SERIES --- CHAP 3
- $k\,dx + \frac{1}{y}dy = 0$ EXACT --- CHAP 1

4. (10 points) Solve the following initial value problem. Use any applicable method.

$$y'' - 5y' + 4y = 2e^{2x}, \qquad y(0) = 3,\ y'(0) = 5$$

Homo. eq'n:

$y'' - 5y' + 4y = 0$

$r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 4,\ r = 1$

$y_c(x) = c_1 e^{4x} + c_2 e^{x}$

Non Homo. eq'n:

$g(x) = 2e^{2x}$

$y_p(x) = Ae^{2x}$

$y_p'' - 5y_p' - 4y_p =$

$4Ae^{2x} - 10Ae^{2x} + 4Ae^{2x}$

$= -2Ae^{2x} = 2e^{2x}$

$\Downarrow$

$A = -1$

$y_p(x) = -e^{2x}$

$y(x) = c_1 e^{4x} + c_2 e^{x} - e^{2x}$

$y(0) = 3 \Rightarrow c_1 + c_2 - 1 = 3$

$y'(0) = 5 \Rightarrow 4c_1 + c_2 - 2 = 5$

$c_1 + c_2 = 4$

$4c_1 + c_2 = 7$

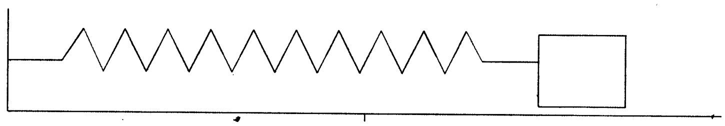
$-3c_1 = -3$

$c_1 = 1$

$c_2 = 3$

$\boxed{y(x) = e^{4x} + 3e^{x} - e^{2x}}$

5. (15 points) A 1-kg mass is attached to a spring with spring constant 5 N/m. The damping constant for the system is 2 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and released from rest. Set up and solve the initial value problem that describes the motion. Write your final answer in terms of a single trigonometric function with phase shift.



$$x'' + 2x' + 5x = 0$$

$$x(0) = 1, \quad x'(0) = 0$$

LAPLACE TRANSFORMS ...

$$s^2 X(s) - s + 2s X(s) - 2 + 5 X(s) = 0$$

$$(s^2 + 2s + 5) X(s) = s + 2$$

$$X(s) = \frac{s+2}{(s+1)^2 + 4} = \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$$

$$x(t) = e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t$$

$c_1 = 1$

$c_2 = \frac{1}{2}$

$$A = \sqrt{1^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\varphi = \tan^{-1} 2$$

$$x(t) = \frac{\sqrt{5}}{2} e^{-t} \sin(2t + \tan^{-1} 2)$$

6. (12 points) Use Laplace transform methods to find a nontrivial solution satisfying $x(0) = 0$. You may use your table of Laplace transforms whenever necessary.

$$tx'' - 2x' + tx = 0$$

$$\mathcal{L}\{tx''\} - 2\,\mathcal{L}\{x'\} + \mathcal{L}\{tx\} = 0$$

$$-\frac{d}{ds}\left(s^2 X(s) - s\,\cancel{x(0)}^{\,0} - \cancel{x'(0)}^{\,\text{CONST}}\right) - 2sX(s) + 2\,\cancel{x(0)}^{\,0} - \frac{d}{ds}X(s) = 0$$

$$-s^2 X'(s) - 2s\,X(s) - 2s\,X(s) - X'(s) = 0$$

$$-(s^2+1)\,X'(s) = 4s\,X(s)$$

$$X'(s) = \frac{-4s}{s^2+1}\,X(s)$$

$$\frac{dX}{X} = \frac{-4s}{s^2+1}\,ds \quad\Rightarrow\quad \ln|X| = -2\ln(s^2+1) + C_1$$

$u = s^2+1$

$du = 2s\,ds$

$$X(s) = \frac{C_2}{(s^2+1)^2} \qquad \text{\#25}$$

$$x(t) = \frac{C}{2}\left(\sin t - t\cos t\right)$$

7. (10 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Rather than derive the solution method (as we did in class), you can use Theorem 1 on page 593.

$$10\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 5, \quad t > 0,$$

$$u(0,t) = u(5,t) = 0,$$

$$u(x,0) = 25$$

$k = \frac{1}{10}$, $L = 5$

THEOREM 1 ...

SOLUTION IS $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/250} \sin\frac{n\pi x}{5},}$

WHERE

$$b_n = \frac{2}{5}\int_0^5 25 \sin\frac{n\pi x}{5}\,dx$$

$$= 10\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= -\frac{50}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^5$$

$$= -\frac{50}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{-50}{n\pi}\left[(-1)^n - 1\right]$$

7 $$\boxed{b_n = \begin{cases} \frac{100}{n\pi}, & n = 1, 3, 5, 7, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}}$$