Math 240 - Homework 2
October 7, 2021

Name key

Score ______

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. This assignment is due October 12.

1. (2 points) Show that $y_1(x) = 1$ and $y_2(x) = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution. Why is a linear combination of two solutions not a solution?

$y_1(x) = 1$

$y_1'(x) = 0$

$y_1''(x) = 0$

$yy'' + (y')^2 = (1)(0) + (0)^2 = 0$ ✓

$y_2(x) = \sqrt{x}, \quad x > 0$

$y_2'(x) = \frac{1}{2\sqrt{x}}$

$y_2''(x) = -\frac{1}{4\sqrt{x^3}}$

$yy'' + (y')^2 = \sqrt{x}\left(-\frac{1}{4\sqrt{x^3}}\right) + \left(\frac{1}{4x}\right)$

$= -\frac{1}{4x} + \frac{1}{4x} = 0$ ✓

$y(x) = 1 + \sqrt{x}$

$y'(x) = \frac{1}{2\sqrt{x}}$

$y''(x) = -\frac{1}{4\sqrt{x^3}}$

$yy'' + (y')^2 = (1+\sqrt{x})\left(-\frac{1}{4\sqrt{x^3}}\right) + \frac{1}{4x}$

$= -\frac{1}{4\sqrt{x^3}} \neq 0$

SINCE THE EQN. IS NOT LINEAR, WE CANNOT EXPECT A SUM OF SOLUTIONS TO BE A SOLUTION.

2. (2 points) Solve the Cauchy-Euler equation $x^2y'' + 2xy' - 12y = 0$.

$x = e^t \Rightarrow \frac{d^2y}{dt^2} + \frac{dy}{dt} - 12y = 0$

$r^2 + r - 12 = 0$

$(r+4)(r-3) = 0$

$y = c_1 e^{-4t} + c_2 e^{3t}$

$\Downarrow$

$$\boxed{y(x) = c_1 x^{-4} + c_2 x^3}$$

Turn over.

$y'(x) = -5c_2 e^{-5x} + c_3 e^{-5x} - 5c_3 x e^{-5x}$

$y''(x) = 25c_2 e^{-5x} - 5c_3 e^{-5x} + 25c_3 x e^{-5x} - 5c_3 e^{-5x}$

3. (2 points) Solve the initial value problem.

$$y''' + 10y'' + 25y' = 0; \quad y(0) = 3,\ y'(0) = 4,\ y''(0) = 5$$

$r^3 + 10r^2 + 25r = 0$

$r(r+5)^2 = 0$

$r = 0,\ r = -5,\ r = -5$

$y(x) = c_1 e^{0x} + c_2 e^{-5x} + c_3 x e^{-5x}$

$y(x) = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}$

$y(0) = 3 \Rightarrow c_1 + c_2 = 3$

$y'(0) = 4 \Rightarrow -5c_2 + c_3 = 4$

$y''(0) = 5 \Rightarrow 25c_2 - 10c_3 = 5$

$-5c_3 = 25$

$c_3 = -5$

$\Downarrow$

$c_2 = -\frac{9}{5}$

$c_1 = \frac{24}{5}$

$y(x) = \frac{24}{5} - \frac{9}{5} e^{-5x} - 5x e^{-5x}$

4. (2 points) Solve the Cauchy-Euler equation $x^2y'' + xy' + 9y = 0$.

$x = e^t \Rightarrow \frac{d^2y}{dt^2} + 9y = 0$

$r^2 + 9 = 0$

$r = 0 \pm 3i \Rightarrow y(t) = c_1 \cos 3t + c_2 \sin 3t$

$y(x) = c_1 \cos(\ln x^3) + c_2 \sin(\ln x^3)$

5. (2 points) The equation of motion of a mass in a mass-spring system satisfies the equation

$$4x'' + 20x' + 169x = 0; \quad x(0) = 4,\ x'(0) = 16.$$

Determine the equation of motion. Write your final answer in terms of a single sine or cosine.

$4r^2 + 20r + 169 = 0$

$r = \frac{-20 \pm \sqrt{400 - 4(4)(169)}}{8}$

$= \frac{-20 \pm \sqrt{-2304}}{8} = \frac{-20 \pm 48i}{8}$

$\alpha = -\frac{5}{2},\ \beta = 6$

$x(t) = c_1 e^{-\frac{5}{2}t} \cos 6t + c_2 e^{-\frac{5}{2}t} \sin 6t$

$x(0) = 4 \Rightarrow c_1 = 4$

$x'(0) = 16 \Rightarrow -\frac{5}{2}c_1 + 6c_2 = 16$

$-10 + 6c_2 = 16$

$c_2 = \frac{13}{3}$

$x(t) = e^{-\frac{5}{2}t}\left(4 \cos 6t + \frac{13}{3} \sin 6t\right)$

$A = \sqrt{16 + \frac{169}{9}} = \frac{\sqrt{313}}{3}$

φ IS IN QUAD I WITH $\tan \varphi = \frac{4}{\frac{13}{3}} = 12/13$

$x(t) = \frac{\sqrt{313}}{3} e^{-\frac{5}{2}t} \sin\left(6t + \tan^{-1}\left(\frac{12}{13}\right)\right)$