Math 240 - Homework 3

December 2, 2021

Name key

Score

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. This assignment is due December 9.

1. (4 points) Use Laplace transform techniques to solve:

$$tx'' + (t-2)x' + x = 0, \quad x(0) = 0.$$

$$\mathcal{L}\{tx''\} + \mathcal{L}\{tx'\} - 2\mathcal{L}\{x'\} + \mathcal{L}\{x\} = 0$$

$$-\frac{d}{ds}\Big(s^2X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{\text{CONST}}\Big) + -\frac{d}{ds}\Big(sX(s) - \cancel{x(0)}^{0}\Big) - 2\Big(sX(s) - \cancel{x(0)}^{0}\Big) + X(s) = 0$$

$$-2sX(s) - s^2X'(s) - sX'(s) - X(s) - 2sX(s) + X(s) = 0$$

$$-(s^2+s)X'(s) + (-4s)X(s) = 0$$

$$\frac{dX}{ds} = \frac{-4}{s+1}X \quad \text{OR} \quad \frac{dX}{X} = -\frac{4}{s+1}ds$$

$$\ln|X| = -4\ln|s+1| + C_1$$

$$X(s) = \frac{C_2}{(s+1)^4} = \frac{C_2}{3!}\frac{3!}{(s+1)^4}$$

$$x(t) = \frac{C_2}{6}t^3e^{-t}$$

$$\boxed{x(t) = C_3t^3e^{-t}}$$

Turn over.

2. (2 points) Use the convolution theorem to find the inverse Laplace transform of $F(s) = \dfrac{s^2}{(s^2+4)^2}$.

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \cdot \frac{s}{s^2+4} \right\} = \cos 2t * \cos 2t = \int_0^t \cos 2\tau \cdot \cos 2(t-\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t \left[\cos(4\tau - 2t) + \cos 2t\right] d\tau = \frac{1}{8}\sin(4\tau - 2t) + \frac{1}{2}\tau\cos 2t \Big|_0^t$$

$$= \frac{1}{8}\sin 2t + \frac{1}{2} t\cos 2t - \frac{1}{8}\sin(-2t)$$

$$= \boxed{\frac{1}{4}\sin 2t + \frac{1}{2} t\cos 2t}$$

3. (2 points) Find the convolution of $f(t) = g(t) = e^{at}$.

$$e^{at} * e^{at} = \int_0^t e^{a\tau} e^{a(t-\tau)}\, d\tau = \int_0^t e^{at}\, d\tau$$

$$= \tau e^{at} \Big|_0^t = \boxed{t e^{at}}$$

4. (2 points) Find the inverse Laplace transform of $F(s) = \ln \dfrac{s-2}{s+2}$.

$$F(s) = \ln(s-2) - \ln(s+2)$$

$$F'(s) = \frac{1}{s-2} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s+2}\right\}$$

$$= -\frac{1}{t}\left(e^{2t} - e^{-2t}\right) = \boxed{\frac{e^{-2t} - e^{2t}}{t}}$$