Math 240 - Quiz 6
October 28, 2021

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. This quiz is due November 4.

1. (4 points) Find the first eight nonzero terms of the power series solution centered at $x = 0$.

$$(1-x)y' - y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(1-x)y' - y =$$

$$\underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{n+1 \to n} - \underbrace{\sum_{n=1}^{\infty} n a_n x^n}_{\text{START AT } n=0} - \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - (n+1) a_n\right] x^n = 0$$

$$(n+1) a_{n+1} = (n+1) a_n$$

$$\Downarrow$$

$$a_{n+1} = a_n; \quad n = 0, 1, 2, 3, \ldots$$

$$\Downarrow$$

$$a_n = a_0; \quad n = 0, 1, 2, 3, \ldots$$

$$\boxed{y(x) = a_0 + a_0 x + a_0 x^2 + a_0 x^3 + a_0 x^4 + a_0 x^5 + a_0 x^6 + a_0 x^7 + \cdots}$$

$$= a_0 \sum_{n=0}^{\infty} x^n$$

2. (1 point) Based on the pattern in your solution above, write the power series solution as an infinite sum in summation notation.

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} x^n} = \frac{a_0}{1-x}, \quad |x| < 1$$

Turn over.

3. (4 points) Find the first eight nonzero terms of the power series solution centered at $x = 0$.

$$y' - x^2 y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - x^2 y =$$

$$\underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{n+1 \to n} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+2}}_{n-2 \to n} =$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n =$$

$$a_1 + 2a_2 x + \sum_{n=2}^{\infty} \left[(n+1) a_{n+1} - a_{n-2} \right] x^n = 0$$

$a_1 = 0$

$a_2 = 0$

a_0 IS ARBITRARY.

$$(n+1) a_{n+1} - a_{n-2} = 0$$

$n = 2, 3, 4, \ldots$

$$a_{n+1} = \frac{1}{n+1} a_{n-2}; \quad n = 2, 3, 4, \ldots$$

$$a_3 = \frac{1}{3} a_0$$

$$a_4 = \frac{1}{4} a_1 = 0$$

$$a_5 = \frac{1}{5} a_2 = 0$$

$$a_6 = \frac{1}{6} a_3 = \frac{1}{6 \cdot 3} a_0$$

$$a_7 = a_8 = 0$$

$$a_9 = \frac{1}{9} a_6 = \frac{1}{9 \cdot 6 \cdot 3} a_0$$

$$a_{10} = a_{11} = 0$$

IN GENERAL,

$$a_{3n} = \frac{1}{3^n n!} a_0$$

4. (1 point) Based on the pattern in your solution above, write the power series solution as an infinite sum in summation notation.

$$y(x) = a_0 \left(1 + \frac{1}{3} x^3 + \frac{1}{18} x^6 + \frac{1}{162} x^9 + \frac{1}{1944} x^{12} + \frac{1}{29160} x^{15} + \frac{1}{524880} x^{18} + \frac{1}{11022480} x^{21} + \cdots \right)$$

$$= \boxed{a_0 \cdot \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!}} = a_0 e^{x^3/3}, \quad x \in \mathbb{R}$$