Math 240 - Quiz 7

November 18, 2021

Name Key

Score

Show all work to receive full credit. Supply explanations when necessary. You must work individually on this quiz. This quiz is due December 2.

1. (2 points) Use the definition of the Laplace transform to find the transform of f.

$$f(t) = \begin{cases} 1, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

$$F(s) = \int_0^3 e^{-st}\,dt + \int_3^\infty 2e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_{t=0}^{t=3} - \frac{2}{s}e^{-st}\Big|_{t=3}^{t\to\infty}$$

$$= -\frac{1}{s}\left(e^{-3s}-1\right) + \frac{2}{s}e^{-3s} - \cancel{\lim_{t\to\infty}\frac{2}{s}e^{-st}} \xrightarrow{} 0,\ s>0$$

$$= \boxed{\frac{1}{s}\left(e^{-3s}+1\right),\quad s>0}$$

2. (2 points) Find the inverse Laplace transform of $F(s)$. You may use technology to compute any required partial fraction decompositions.

$$F(s) = \frac{s+1}{s^2(s+2)^3}$$

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{(s+2)^3}$$

$$A = -\frac{1}{16},\ B = \frac{1}{8},\ C = \frac{1}{16},\ D = 0,\ E = -\frac{1}{4}$$

$$F(s) = -\frac{1}{16}\frac{1}{s} + \frac{1}{8}\frac{1}{s^2} + \frac{1}{16}\frac{1}{s+2} - \frac{1}{8}\frac{2}{(s+2)^3}$$

$$\boxed{f(t) = -\frac{1}{16} + \frac{1}{8}t + \frac{1}{16}e^{-2t} - \frac{1}{8}t^2e^{-2t}}$$

Turn over.

3. (3 points) Use Laplace transform techniques to solve the initial value problem. You may use technology to compute any required partial fraction decompositions.

$$y'' + 16y = \sin t, \quad y(0) = 0,\ y'(0) = 1$$

$$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$$

$$s^2 Y(s) - s y(0) - y'(0) + 16 Y(s) = \frac{1}{s^2+1}$$

$$s^2 Y(s) - 1 + 16 Y(s) = \frac{1}{s^2+1}$$

$$(s^2+16) Y(s) = \frac{1}{s^2+1} + 1$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+16)} + \frac{1}{s^2+16}$$

PFD...

$$Y(s) = \frac{1/15}{s^2+1} + \frac{14/15}{s^2+16}$$

$$= \frac{1}{15}\frac{1}{s^2+1} + \frac{7}{30}\frac{4}{s^2+16}.$$

$$\boxed{y(t) = \frac{1}{15}\sin t + \frac{7}{30}\sin 4t}$$

4. (3 points) Use Laplace transform techniques to solve the initial value problem. You may use technology to compute any required partial fraction decompositions.

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2,\ y'(0) = 6$$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$s^2 Y(s) - s y(0) - y'(0) - 6s Y(s) + 6y(0) + 9 Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9) Y(s) - 2s + 6 = \frac{2}{(s-3)^3}$$

$$(s-3)^2 Y(s) = \frac{2}{(s-3)^3} + 2s - 6$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3} = \frac{1}{12}\left(\frac{4!}{(s-3)^5}\right) + 2\left(\frac{1}{s-3}\right)$$

$$\boxed{y(t) = \frac{1}{12} t^4 e^{3t} + 2e^{3t}}$$