Math 240 - Test 1
September 16, 2021

Name Key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $(x^2 + y^2)\,dx + 2xy\,dy = 3\,dx$

   1ST ORDER, NONLINEAR, ORDINARY

   (b) $\dfrac{\partial^2 u}{\partial y^2} - xy^2 u = 6y^2 - \sin x$

   2ND ORDER, LINEAR, PARTIAL

   (c) $z'' - 4z' + 3z = (t^2 + 5)\sin t$

   2ND ORDER, LINEAR, ORDINARY

   (d) $yy''' + xy'' = xe^x$

   3RD ORDER, NONLINEAR, ORDINARY

2. (5 points) Suppose you are sketching the direction field for the differential equation

$$x^2 \frac{dy}{dx} + 3xy^3 = 4.$$

   (a) What is the slope of the solution curve passing through $(2, 3)$?

$$\frac{dy}{dx} = \frac{4 - 3xy^3}{x^2} \qquad \left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = \frac{4 - 3(2)(27)}{4} = -39.5$$

   (b) Find a point through which you would not expect a solution curve to exist. Say why.

   $\frac{dy}{dx}$ DOES NOT EXIST AT ANY POINT WHERE $X = 0$,

   FOR EXAMPLE $(0,0)$.

   (NOTICE THAT $(0,0)$ CANNOT SATISFY THE EQUATION FOR ANY FUNCTION y.)

3. (8 points) Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{10}{x^2+1}, \quad y(0) = 0$$

$$y(x) = \int \frac{10}{x^2+1}\,dx$$

$$= 10 \tan^{-1} x + C.$$

$$y(0) = 0 \Rightarrow 0 = 10 \tan^{-1} 0 + C$$

$$0 = 0 + C$$

$$C = 0$$

$$\boxed{y(x) = 10 \tan^{-1} x}$$

4. (12 points) Analyze each initial value problem and determine whether we should expect a unique solution, more than one solution, or no solution to exist through the given point.

(a) $x\dfrac{dy}{dx} = (2y-10)^{1/3}, \quad y(4) = 5$

$f(x,y) = \dfrac{(2y-10)^{1/3}}{x}$ CONTINUOUS AROUND $(4,5)$

$f_y(x,y) = \dfrac{2}{3x}(2y-10)^{-2/3}$ NOT DEFINED AT $y=5$

WE EXPECT A SOLUTION BUT NOT NECESSARILY A SINGLE SOLUTION.

(b) $\dfrac{dy}{dx} - x^2 y = \sin^3 x, \quad y(\pi) = 2$

THIS EQUATION IS LINEAR, AND ITS COEFFICIENTS ARE CONTINUOUS EVERYWHERE. $\Rightarrow$ UNIQUE SOLUTION THROUGH ANY POINT.

(c) $y\dfrac{dy}{dx} = e^x, \quad y(1) = 0$

$f(x,y) = \dfrac{e^x}{y}$ NOT DEFINED WHERE $y = 0$.

WE DO NOT EXPECT A SOLUTION THROUGH $(1,0)$.

5. (3 points) Without solving the following differential equation, show that it is NOT exact.

$$\underbrace{(\cos x \sin x - xy^2)}_{M(x,y)}\,dx + \underbrace{y(1+x^2)}_{N(x,y)}\,dy = 0$$

$$\frac{\partial M}{\partial y} = -2xy \neq \frac{\partial N}{\partial x} = 2xy \Rightarrow \text{NOT EXACT.}$$

6. (12 points) Solve the following initial value problem:

$$y' + 2xy = x, \quad y(0) = -2$$

LINEAR, 1ST-ORDER

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$\mu(x)\,y(x) = \int \mu(x)\,q(x)\,dx$$

$$\Downarrow$$

$$e^{x^2} y(x) = \int x e^{x^2}\,dx = \frac{1}{2}\int e^u\,du$$

$$u = x^2 \qquad = \frac{1}{2}e^u + C$$

$$du = 2x\,dx$$

$$e^{x^2} y(x) = \frac{1}{2}e^{x^2} + C$$

$$y(x) = \frac{1}{2} + Ce^{-x^2}$$

$$y(0) = -2$$

$$\Downarrow$$

$$-2 = \frac{1}{2} + C$$

$$\Downarrow$$

$$C = -\frac{5}{2}$$

$$\boxed{y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}}$$

7. (12 points) Solve the following initial value problem:

$$x^2\frac{dy}{dx} = (1-x^2)y^3, \quad y(1) = 1$$

SEPARABLE

$$\frac{dy}{y^3} = \frac{1-x^2}{x^2}\,dx$$

$$\frac{1}{y^3}\,dy = \left(\frac{1}{x^2} - 1\right)dx$$

$$-\frac{1}{2}\frac{1}{y^2} = -\frac{1}{x} - x + C$$

$$y(1) = 1 \Rightarrow -\frac{1}{2} = -2 + C$$

$$C = \frac{3}{2}$$

$$-\frac{1}{2}\frac{1}{y^2} = -\frac{1}{x} - x + \frac{3}{2}$$

$$\frac{1}{y^2} = \frac{2}{x} + 2x - 3$$

$$y^2 = \frac{1}{\frac{2}{x} + 2x - 3} = \frac{x}{2 + 2x^2 - 3x}$$

$$\boxed{y(x) = \sqrt{\frac{x}{2 + 2x^2 - 3x}}}$$

★ POSITIVE SQUARE ROOT TO COINCIDE WITH INITIAL CONDITION.

8. (18 points) Consider the following initial value problem:

$$(e^x y + xe^x y)\, dx + (xe^x + 2)\, dy = 0, \quad y(0) = 4.$$

(a) Use the test for exactness to show that the DE is exact?

$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x} = xe^x + e^x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ✓.

(b) Solve the initial value problem.

$\frac{\partial F}{\partial x} = e^x y + xe^x y \Rightarrow F(x,y) = \cancel{e^x y} + (xe^x - e^x)y + g(y)$

$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x$

$u = x \quad du = dx$
$dv = e^x\, dx \quad v = e^x$

$\frac{\partial F}{\partial y} = xe^x + 2 \Rightarrow F(x,y) = xe^x y + 2y + h(x)$

$F(x,y) = xye^x + 2y = C \quad y(0) = 4 \Rightarrow C = 8$

THAT SOL'N IS IMPLICIT.

$xye^x + 2y = 8$

(c) Is your solution explicit or implicit?

My FINAL SOLUTION IS EXPLICIT.

SOLVE FOR y:

$y = \frac{8}{xe^x + 2}$

(d) Show that the equation is also separable.

$\frac{dy}{dx} = \frac{-(e^x y + xe^x y)}{xe^x + 2}$ FACTOR OUT y.

$\frac{dy}{dx} = \left[-\frac{(e^x + xe^x)}{(xe^x + 2)} \right] [y] = f(x)\, g(y)$

(e) Is the equation also linear?

Yes!

$\frac{dy}{dx} + \left(\frac{e^x + xe^x}{xe^x + 2} \right) y = 0$

$p(x) = \frac{e^x + xe^x}{xe^x + 2}, \quad q(x) = 0$

The following problems make up the take-home portion of the test. These problems are due September 21, 2021. You must work on your own.

9. (8 points) The US population from 1790 to 1940 can be approximated by the solution of the initial value problem

$$\frac{dP}{dt} = 0.0318P - 0.000170P^2, \quad P(0) = 3.9,$$

IT'S BERNOULLI OR SEPARABLE.

where P is in millions and t is in years since 1790.

(a) Solve for $P(t)$.

LET $a = 0.0318$ & $b = 0.000170$

$$\frac{dP}{P(a-bP)} = dt$$

$$\frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP}$$

$$1 = A(a-bP) + BP$$

$$Aa = 1 \Rightarrow A = \frac{1}{a}$$

$$B - Ab = 0 \Rightarrow B = Ab = \frac{b}{a}$$

$$\int\left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right)dP = \int dt$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + C_1$$

$$\ln\left|\frac{P}{a-bP}\right| = at + C_2$$

$$\frac{P}{a-bP} = \pm e^{at+C_2} = C_3 e^{at}$$

$$P(0) = 3.9 \Rightarrow C_3 = \frac{3.9}{a - 3.9b} \approx 125.253$$

$$P = C_3 e^{at}(a-bP) = C_3 a e^{at} - C_3 b e^{at} P$$

$$P(t) = \frac{C_3 a e^{at}}{1 + C_3 b e^{at}} \Rightarrow \boxed{P(t) = \frac{3.983 e^{0.0318t}}{1 + 0.0213 e^{0.318t}}}$$

(b) If the actual population in 1900 was 76.0 million, find the percent error in the approximation given by P.

110 YEARS FROM 1790

$$P(110) = \frac{3.983 e^{3.498}}{1 + 0.0213 e^{3.498}} \approx 77.25$$

$$\left|\frac{76 - 77.25}{76}\right| \times 100\% \approx \boxed{1.6\%}$$

10. (12 points) A tank initially contains 40 gal of pure water. A salt solution containing 3 lb of salt per gallon enters the tank at 2 gal/min and is uniformly mixed. The mixed solution leaves the tank at 3 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. How much salt is in the tank when the volume is 20 gal? When is the amount of salt in the tank greatest?

$3 \text{ lb/GAL} \times 2 \text{ GAL/MIN} = 6 \text{ lb/min}$

2 GAL/MIN IN & 3 GAL/MIN OUT

$\Downarrow$

$V(t) = 40 - t, \quad 0 \le t \le 40$

$V(0) = 40$

$A(0) = 0$

$3 \text{ GAL/min} \times \dfrac{A(t)}{V(t)} \text{ lb/GAL}$

$\dfrac{3A}{40-t} \text{ lb/MIN}$

$$\frac{dA}{dt} = 6 - \frac{3A}{40-t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{3}{40-t} A = 6$$

$$\mu(t) := e^{\int \frac{3}{40-t}\,dt} = e^{-3\ln|40-t|} = |40-t|^{-3} = \frac{1}{(40-t)^3}, \quad 0 \le t \le 40$$

$$\frac{1}{(40-t)^3} A(t) = \int 6(40-t)^{-3}\,dt = 3(40-t)^{-2} + C$$

$$A(t) = 3(40-t) + C(40-t)^3$$

$$A(0) = 0 \Rightarrow 0 = 120 + C\,40^3$$

$$C = -\frac{120}{64000} = -\frac{3}{1600}$$

$$\boxed{A(t) = 3(40-t) - \frac{3}{1600}(40-t)^3}$$

$V(t) = 20$ WHEN $t = 20$

$$\boxed{A(20) = 45 \text{ lb}}$$

CONTINUED ON NEXT PAGE...

MAX AMOUNT OF SALT ?

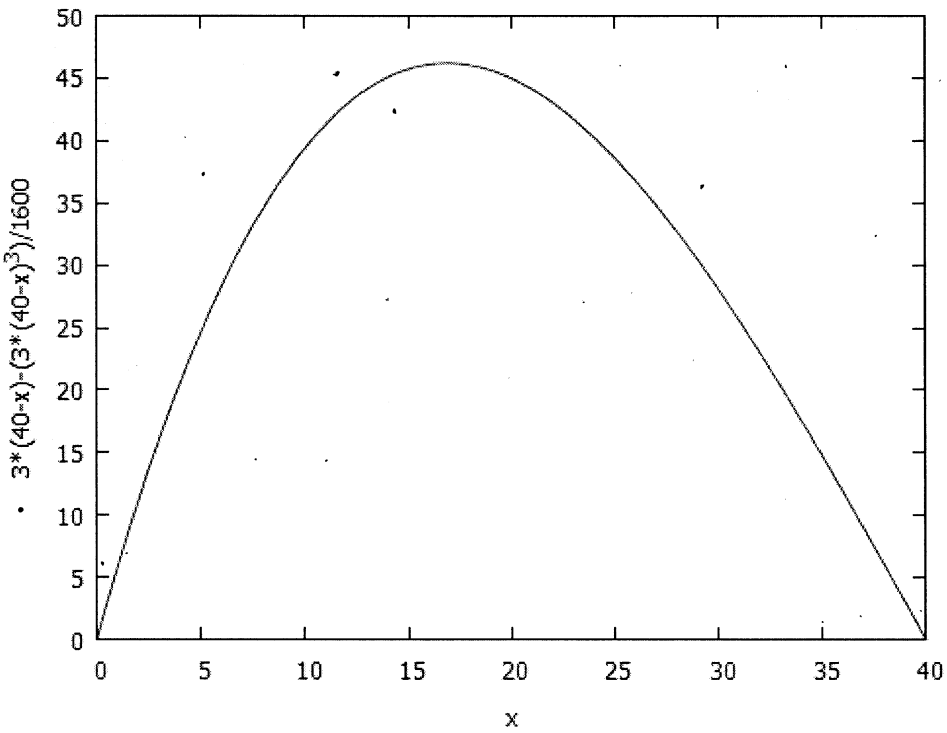
$$A'(t) = -3 + \frac{9}{1600}(40-t)^2$$

$$A'(t) = 0 \Rightarrow (40-t)^2 = \frac{4800}{9}$$

$$40-t = \sqrt{\frac{4800}{9}}$$

$$t = 40 - \sqrt{\frac{4800}{9}} \approx \boxed{16.906 \text{ MIN}}$$

SEE ATTACHED GRAPH.



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