# Math 240 - Test 2

October 14, 2021

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (12 points) Solve: $\dfrac{dy}{dx} = \dfrac{y^4 - x^4}{2xy^3}$

$$\frac{dy}{dx} = \frac{y}{2x} - \frac{x^3}{2y^3}$$

OR

$$2\frac{dy}{dx} = \frac{y}{x} - \left(\frac{x}{y}\right)^3$$

LET $u = \dfrac{y}{x}$ OR $ux = y$

$$u + x\frac{du}{dx} = \frac{dy}{dx}$$

$$2u + 2x\frac{du}{dx} = u - \frac{1}{u^3}$$

$$2x\frac{du}{dx} = -u - \frac{1}{u^3}$$

$$-2x\frac{du}{dx} = \frac{u^4+1}{u^3}$$

$$\frac{2u^3}{u^4+1}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{2u^3}{u^4+1}\,du = \frac{1}{2}\ln|u^4+1|$$

$w = u^4 + 1$

$dw = 4u^3\,du$

$$\frac{1}{2}\ln|u^4+1| = -\ln|x| + C_1$$

$$\ln\sqrt{u^4+1} = \ln\frac{1}{|x|} + C_1$$

EXPONENTIATE EACH SIDE ...

$$\sqrt{u^4+1} = \frac{C_2}{|x|}$$

$$u^4 + 1 = \frac{C_3}{x^2}$$

$$\left(\frac{y}{x}\right)^4 = \frac{C_3}{x^2} - 1$$

$$y^4 = Cx_2 - x^4$$

OR

$$\boxed{y(x) = \sqrt[4]{Cx_2 - x^4}}$$

2. (12 points) Solve: $\frac{dy}{dx} - y = e^x y^2$

$$y^{-2}\frac{dy}{dx} - y^{-1} = e^x$$

$$u = y^{-1} \qquad \frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-\frac{du}{dx} - u = e^x$$

$$\frac{du}{dx} + u = -e^x$$

$$\mu(x) = e^{\int 1\,dx} = e^x$$

$$\mu(x)\, u(x) = \int \mu(x)(-e^x)\,dx$$

$$e^x u(x) = \int -e^{2x}\,dx = -\frac{1}{2}e^{2x} + C$$

$$u(x) = -\frac{1}{2}e^x + Ce^{-x}$$

$$\frac{1}{y(x)} = -\frac{1}{2}e^x + Ce^{-x}$$

$$y(x) = \frac{1}{-\frac{1}{2}e^x + Ce^{-x}} \quad \text{OR} \quad \boxed{y(x) = \frac{-2}{e^x + Ce^{-x}}}$$

$y(x) = 0$ IS ALSO A SOLUTION

3. (15 points) Consider the equation $xy'' + 5y' = 0,\ x > 0$.

(a) Verify that $y_1(x) = 1$ and $y_2(x) = \dfrac{1}{x^4}$ are solutions.

$y(x) = 1 \ldots$

$y'(x) = 0$

$y''(x) = 0$

$(x)(0) + 5(0) = 0$

$0 = 0$ ✓

$y(x) = x^{-4} \ldots$

$y'(x) = -4x^{-5}$

$y''(x) = 20x^{-6}$

$(x)(20x^{-6}) + 5(-4x^{-5})$

$= 20x^{-5} - 20x^{-5} = 0$

$0 = 0$ ✓

(b) Use the Wronskian to show that $y_1$ and $y_2$ are linearly independent on $(0, \infty)$.

$$\begin{vmatrix} 1 & x^{-4} \\ 0 & -4x^{-5} \end{vmatrix} = -4x^{-5} = \frac{-4}{x^5} \neq 0 \text{ For } x > 0$$

$\Rightarrow y_1$ & $y_2$ ARE LIN. INDEP.

(c) Now consider the nonhomogeneous equation $xy'' + 5y' = 12x,\ x > 0$. Verify that $y_p(x) = x^2$ is a solution.

$y(x) = x^2 \ldots$

$y'(x) = 2x$

$y''(x) = 2$

$x(2) + 5(2x) =$

$2x + 10x = 12x$

$12x = 12x$ ✓

(d) Use what you've learned in parts (a), (b), and (c) to find the solution of the IVP $xy'' + 5y' = 12x;\ y(1) = 2,\ y'(1) = 10$.

$y(x) = C_1 + C_2x^{-4} + x^2$

$y(1) = C_1 + C_2 + 1 = 2$

$C_1 + C_2 = 1$

$y'(x) = -4C_2x^{-5} + 2x$

$y'(1) = -4C_2 + 2 = 10$

$-4C_2 = 8$

$C_2 = -2$

$C_1 = 3$

$\boxed{y(x) = 3 - 2x^{-4} + x^2}$

Yes!

(e) Is your solution in part (d) unique? Explain.

$y'' + \frac{5}{x}y' = 12$

THIS IS A LINEAR EQUATION IN STANDARD FORM WHOSE COEFFICIENT FUNCTIONS ARE CONTINUOUS FOR $x > 0$.

IT FOLLOWS FROM OUR EXISTENCE/UNIQUENESS THEOREM. THAT THE EQUATION HAS A UNIQUE SOLUTION FOR $x > 0$ FOR ANY INITIAL CONDITIONS.

4. (8 points) Solve the following initial value problem.

$$y'' + y' - 12y = 0; \quad y(0) = 3,\ y'(0) = 23$$

CHAR. eqn...

$r^2 + r - 12 = 0$

$(r+4)(r-3) = 0$

$r = -4,\ r = 3$

$y_1(x) = e^{-4x}$

$y_2(x) = e^{3x}$

$y(x) = c_1 e^{-4x} + c_2 e^{3x}$

$y'(x) = -4c_1 e^{-4x} + 3c_2 e^{3x}$

$y(0) = c_1 + c_2 = 3$

$y'(0) = -4c_1 + 3c_2 = 23$

$7c_2 = 35 \Rightarrow c_2 = 5$

$\Rightarrow c_1 = -2$

$y(x) = -2e^{-4x} + 5e^{3x}$

5. (8 points) Find the general solution: $y^{(5)} + 2y^{(3)} = 0$

CHAR. eqn...

$r^5 + 2r^3 = 0$

$r^3(r^2 + 2) = 0$

$r = 0, 0, 0, \sqrt{2}\,i, -\sqrt{2}\,i$

$y_1(x) = e^{0x} = 1$

$y_2(x) = xe^{0x} = x$

$y_3(x) = x^2 e^{0x} = x^2$

$y_4(x) = e^{0x} \cos \sqrt{2}\,x$

$y_5(x) = e^{0x} \sin \sqrt{2}\,x$

$y(x) = C_1 + C_2 x + C_3 x^3 + C_4 \cos \sqrt{2}\,x + C_5 \sin \sqrt{2}\,x$

6. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion, underdamped motion, overdamped motion,* or *critically damped motion.* Match each equation with the corresponding situation.

(a) $x(t) = \underbrace{6e^{-t/3}} \sin(\sqrt{5}t + \frac{1}{2})$

$\boxed{\text{UNDERDAMPED.}}$ $\quad \alpha = -\frac{1}{3}, \ \beta = \sqrt{5}$

(b) $x(t) = 7e^{-3t} - 9te^{-3t}$

$\boxed{\text{CRITICALLY DAMPED.}}$ $\quad b^2 - 4mk = 0$ TO GET THIS SOLUTION

(c) $5x'' + 8x' + 2x = 0$

$b^2 - 4mk = 64 - 4(5)(2) > 0 \Rightarrow \boxed{\text{OVERDAMPED.}}$

(d) $x'' + 8x = 0$

$b = 0 \quad \boxed{\text{SIMPLE HARMONIC MOTION.}}$

7. (5 points) Let $y_1(x) = x + 1$ and $y_2(x) = x^2 - (x+2)^2$. Compute the Wronskian of $y_1$ and $y_2$. Briefly explain why $y(x) = c_1y_1(x) + c_2y_2(x)$ cannot be the general solution of a 2nd-order, linear, homogeneous differential equation.

$x^2 - x^2 - 4x - 4$

$$W = \begin{vmatrix} x+1 & x^2 - (x+2)^2 \\ 1 & 2x - 2(x+2) \end{vmatrix} = \begin{vmatrix} x+1 & -4x-4 \\ 1 & -4 \end{vmatrix}$$

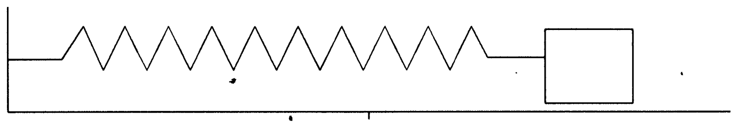
$$= -4(x+1) - (1)(-4x-4)$$

$$= -4x - 4 + 4x + 4 = 0$$

$y_1$ & $y_2$ ARE LINEARLY DEPENDENT.

TWO LINEARLY <u>INDEPENDENT</u> SOLUTIONS ARE REQUIRED FOR A GENERAL SOLUTION.

8. (12 points) A 1-kg mass is attached to a spring with spring constant $\frac{5}{4}$ N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and pushed to the right at $\frac{1}{2}$ m/sec. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.




$x'' + x' + \frac{5}{4}x = 0; \quad x(0) = 1, \ x'(0) = \frac{1}{2}$

$r^2 + r + \frac{5}{4} = 0$

$r^2 + r + \frac{1}{4} = 1$

$\left(r + \frac{1}{2}\right)^2 = 1$

$r = -\frac{1}{2} \pm i$

$\alpha = -\frac{1}{2}, \ \beta = 1$

$x(t) = c_1 e^{-\frac{1}{2}t} \cos t + c_2 e^{-\frac{1}{2}t} \sin t$

$x(0) = 1 \Rightarrow \quad c_1 = 1$

$x'(t) = -\frac{1}{2}c_1 e^{-\frac{1}{2}t} \cos t - c_1 e^{-\frac{1}{2}t} \sin t$
$\quad - \frac{1}{2}c_2 e^{-\frac{1}{2}t} \sin t + c_2 e^{-\frac{1}{2}t} \cos t$

$x'(0) = -\frac{1}{2}c_1 + c_2 = \frac{1}{2}$

$-\frac{1}{2} + c_2 = \frac{1}{2}$

$c_2 = 1$

$x(t) = e^{-\frac{1}{2}t} \cos t + e^{-\frac{1}{2}t} \sin t$

$x(t) = A e^{-\frac{1}{2}t} \sin(t + \varphi)$

$A \sin \varphi = 1$
$A \cos \varphi = 1$ $\Big\} \therefore \varphi$ in Quad I

$\Downarrow$

$A = \sqrt{2}, \quad \varphi = \tan^{-1} 1 = \frac{\pi}{4}$

$$\boxed{x(t) = \sqrt{2}\, e^{-\frac{1}{2}t} \sin\left(t + \frac{\pi}{4}\right)}$$

The following problems make up the take-home portion of the test. These problems are due October 19, 2021. You must work on your own.

9. (8 points) Solve the following initial value problem.

$$y'' = y' \cdot e^y; \quad y(0) = 0,\ y'(0) = 1$$

LET $u = y'$ SO THAT $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$

$u\frac{du}{dy} = ue^y$ OR $\frac{du}{dy} = e^y$ OR $u = 0$

$u=0$ DOES NOT LEAD TO A SOL'N THAT SATISFIES IC'S.

$du = e^y\,dy \Rightarrow u = e^y + C$

$y' = e^y + C$

$y'(0) = 1 \Rightarrow C = 0$

$y' = e^y$

$e^{-y}\,dy = dx$

$-e^{-y} = x + C$

$y(0) = 0 \Rightarrow -e^0 = 0 + C$

$-1 = C$

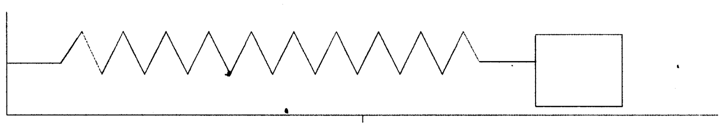
$-e^{-y} = x - 1$

$e^{-y} = 1 - x$

$-y = \ln(1-x)$

$$\boxed{y(x) = -\ln(1-x)}$$

10. (12 points) A 9-kg mass is attached to a spring with spring constant 37 N/m. The damping constant for the system is 6 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and pushed to the left at 2 m/sec. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift. When does the mass pass through equilibrium for the second time?



$x = 0$
(Equilibrium)

$$9x'' + 6x' + 37x = 0; \quad x(0) = 1, \; x'(0) = -2$$

CHAR. EQN:

$$9r^2 + 6r + 37 = 0$$

$$9r^2 + 6r + 1 = -36$$

$$(3r+1)^2 = -36$$

$$3r + 1 = \pm 6i$$

$$r = -\frac{1}{3} \pm 2i$$

$$x(t) = c_1 e^{-\frac{1}{3}t} \cos 2t + c_2 e^{-\frac{1}{3}t} \sin 2t$$

$$x(0) = 1 \Rightarrow c_1 = 1$$

$$x'(0) = -2 \Rightarrow -\frac{1}{3}c_1 + 2c_2 = -2$$

$$-\frac{1}{3} + 2c_2 = -2$$

$$2c_2 = -\frac{5}{3}$$

$$c_2 = -\frac{5}{6}$$

$$x(t) = e^{-\frac{1}{3}t}\left(\cos 2t - \frac{5}{6}\sin 2t\right)$$

$$A = \sqrt{1 + \frac{25}{36}} = \frac{\sqrt{61}}{6}$$

$$\tan \varphi = \frac{1}{-5/6} = -\frac{6}{5}$$

AND $\varphi$ IS IN QUAD 2.

$$\Downarrow$$

$$\varphi = \tan^{-1}\left(-\frac{6}{5}\right) + \pi$$

$$\boxed{x(t) = \frac{\sqrt{61}}{6} e^{-\frac{1}{3}t} \sin\left(2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right)\right)}$$

EQUILIBRIUM WHEN

$$2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right) = k\pi$$

$$t = \frac{(k-1)\pi - \tan^{-1}\left(-\frac{6}{5}\right)}{2}$$

1ST TIME WHEN $k=1$: $t \approx 0.438\text{s}$

$\boxed{\text{2ND TIME WHEN } k=2: \; t \approx 2.009\text{ s}}$