

**Math 240 - Test 3**  
November 11, 2021

Name key Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations where necessary. All integration must be done by hand, unless otherwise specified. You must work individually on this test. The test is due November 16.

1. (16 points) Use undetermined coefficients to solve the following equation.

$$2y'' + 6y' - 20y = 60 \sin 2x$$

DIVIDE BY 2 TO GET

$$y'' + 3y' - 10y = 30 \sin 2x$$

Homo. eqn:

$$y'' + 3y' - 10y = 0$$

$$r^2 + 3r - 10 = 0$$

$$(r+5)(r-2) = 0$$

$$r = -5, r = 2$$

$$y_h(x) = c_1 e^{-5x} + c_2 e^{2x}$$

Non-Homo. eqn:

$$g(x) = 30 \sin 2x$$

$$y_p(x) = A \cos 2x + B \sin 2x$$

$$y_p'(x) = -2A \sin 2x + 2B \cos 2x$$

$$y_p''(x) = -4A \cos 2x - 4B \sin 2x$$

$$y_p'' + 3y_p' - 10y_p = 30 \sin 2x$$

↓

$$(-4A + 6B - 10A) \cos 2x$$

$$+ (-4B - 6A - 10B) \sin 2x$$

$$= 30 \sin 2x$$

↓

$$-14A + 6B = 0 \rightarrow A = \frac{6}{14} B = \frac{3}{7} B$$

$$-6A - 14B = 30$$

$$-\frac{9}{7} B - 14B = 15$$

$$-58B = 105$$

$$B = -\frac{105}{58}$$

$$A = -\frac{45}{58}$$

$$y_p(x) = -\frac{45}{58} \cos 2x - \frac{105}{58} \sin 2x$$

$$y(x) = c_1 e^{-5x} + c_2 e^{2x} - \frac{45}{58} \cos 2x$$

$$- \frac{105}{58} \sin 2x$$

2. (16 points) Use variation of parameters to solve the following equation.

$$y'' + 3y' + 2y = \sin(e^x)$$

Homo. eqn:

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r = -1, r = -2$$

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$$

NonHomo. eqn:

$$g(x) = \sin(e^x)$$

$$y_1(x) = e^{-x}, y_2(x) = e^{-2x}$$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -e^{-3x}$$

$$v_1(x) = \int \frac{-e^{-2x} \sin(e^x)}{-e^{-3x}} dx$$

$$= \int e^x \sin(e^x) dx$$

$$= \int \sin u du \quad (u=e^x)$$

$$= -\cos u = -\cos(e^x)$$

$$v_2(x) = \int \frac{e^{-x} \sin(e^x)}{-e^{-3x}} dx$$

$$= \int -e^{2x} \sin(e^x) dx$$

(u = e^x)

$$= -\int u \sin u du$$

+	u	sin u
-	1	-cos u
+	0	-sin u

$$= u \cos u - \sin u$$

$$= e^x \cos(e^x) - \sin(e^x)$$

$$y_p(x) = y_1(x)v_1(x) + y_2(x)v_2(x)$$

$$= -\cancel{e^{-x} \cos(e^x)} + \cancel{e^{-x} \cos(e^x)} - e^{-2x} \sin(e^x)$$

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin(e^x)$$

3. (8 points) Consider the following equation:

$$y'' - 10y' + 25y = 5x^2 e^{5x}.$$

Solve the corresponding homogeneous equation. Then use your table to find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

Homo. eqn:

$$y'' - 10y' + 25y = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r-5)(r-5) = 0$$

$$y_h(x) = c_1 e^{5x} + c_2 x e^{5x}$$

Non-Homo. Eqn:

$$g(x) = 5x^2 e^{5x}$$

$$y_p(x) = x^s (Ax^2 + Bx + C) e^{5x}$$

MUST CHOOSE  $s = 2$

SO THAT

$$y_p(x) = (Ax^4 + Bx^3 + Cx^2) e^{5x}$$

4. (8 points) Use the definition of the Laplace transform to find the transform of  $f(t) = te^{-t}$ . Show all work.

$$F(s) = \int_0^{\infty} e^{-st} (te^{-t}) dt = \int_0^{\infty} t e^{-(s+1)t} dt = \left[ \frac{t}{s+1} + \frac{1}{(s+1)^2} \right] e^{-(s+1)t} \Bigg|_0^{\infty}$$

$$\int t e^{-(s+1)t} dt = \left[ -\frac{t}{s+1} - \frac{1}{(s+1)^2} \right] e^{-(s+1)t}$$

+	t	$e^{-(s+1)t}$
-	1	$-\frac{1}{s+1} e^{-(s+1)t}$
+	0	$\frac{1}{(s+1)^2} e^{-(s+1)t}$

$$\frac{1}{(s+1)^2} - \lim_{t \rightarrow \infty} \frac{\frac{t}{s+1} + \frac{1}{(s+1)^2}}{e^{(s+1)t}}$$

L'HOPITAL'S RULE.

LIMIT IS ZERO.

$$F(s) = \frac{1}{(s+1)^2}$$

5. (16 points) State the recurrence relation that describes the coefficients of the power series solution (centered at  $x = 0$ ), and state the guaranteed (by our theorem) radius of convergence.

$$(x^2 - 1)y'' + 2xy' + 2xy = 0$$

$$y'' + \frac{2x}{x^2-1}y' + \frac{2x}{x^2-1}y = 0$$

ANALYTIC EVERYWHERE EXCEPT WHEN  $x^2 = 1$ , i.e.,  $x = \pm 1$

THIS DISTANCE FROM  $x=0$  TO  $x=\pm 1$  IS ONE UNIT.

THE RADIUS OF CONVERGENCE IS AT LEAST 1.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$-2a_2 + \sum_{n=1}^{\infty} [n(n-1)a_n - (n+2)(n+1)a_{n+2} + 2na_n + 2a_{n-1}] x^n = 0$$

↓

$$a_2 = 0$$

$a_0$  AND  $a_1$  ARE ARBITRARY.

$$a_{n+2} = \frac{(n^2+n)a_n + 2a_{n-1}}{(n+2)(n+1)}$$

$$n = 1, 2, 3, \dots$$

$$(x^2-1)y'' + 2xy' + 2xy =$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1} =$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

6. (16 points) Find a power series solution centered at  $x = 0$ . Then refer to the Common Infinite Series sheet (available on the Lecture Resources page) to find a familiar expression for your solution and its interval of convergence.

$$(x-2)y' + y = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$a_0$  IS ARBITRARY

$$a_{n+1} = \frac{1}{2} a_n; \quad n=0, 1, 2, \dots$$

$$(x-2)y' + y =$$

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} 2n a_n x^{n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n$$

CAN START AT  
 $n=0$  w/  
 NO CHANGE

$$+ \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} [n a_n - 2(n+1) a_{n+1} + a_n] x^n = 0$$

$$(n+1) a_n - 2(n+1) a_{n+1} = 0;$$

$$n=0, 1, 2, 3, \dots$$

$$y(x) = a_0 \left( 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n$$

$$= \frac{a_0}{1 - \frac{x}{2}}, \quad \left| \frac{x}{2} \right| < 1$$

$$= \frac{2a_0}{2-x}, \quad |x| < 2$$

7. (6 points) Use your table of Laplace transforms to determine the transform of each function. Show how you got your answer.

(a)  $f(t) = 2 + 5e^{2t} - \cos 4t$

$$F(s) = 2 \mathcal{L}\{1\} + 5 \mathcal{L}\{e^{2t}\} - \mathcal{L}\{\cos 4t\} = \frac{2}{s} + \frac{5}{s-2} - \frac{s}{s^2+16}$$

#1                      #11                      #8

(b)  $f(t) = t^{3/2} - t \sin 2t$

$$F(s) = \mathcal{L}\{t^{3/2}\} - \mathcal{L}\{t \sin 2t\} = \frac{\Gamma(\frac{5}{2})}{s^{5/2}} - \frac{4s}{(s^2+4)^2}, \quad \Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

#6                      #22

8. (9 points) Find the inverse Laplace transform of each function. Show how you got your answer.

(a)  $F(s) = \frac{3s+1}{s^2+4} = 3 \left( \frac{s}{s^2+4} \right) + \frac{1}{2} \left( \frac{2}{s^2+4} \right)$

#8                      #7

$$f(t) = 3 \cos 2t + \frac{1}{2} \sin 2t$$

(b)  $F(s) = 5s^{-1}e^{-3s}$

$$F(s) = 5 \frac{e^{-3s}}{s} \quad \#54 \Rightarrow f(t) = 5 u(t-3)$$

WHERE  $u(t)$  IS THE UNIT STEP FUNCTION

(c)  $F(s) = \frac{3}{s} - \frac{2}{s^4} - \frac{8}{6-s} = 3 \left( \frac{1}{s} \right) - \frac{2}{3!} \left( \frac{3!}{s^4} \right) + 8 \left( \frac{1}{s-6} \right)$

#1                      #3                      #11

$$f(t) = 3 - \frac{1}{3} t^3 + 8e^{6t}$$

9. (5 points) Consider the differential equation  $x^2 y'' - 3xy' + 8y = 0$ . Explain why it might not be wise to look for a power series solution centered at  $x = 0$ . Then say how you would solve the equation (but do not solve it).

$$y'' - \frac{3}{x} y' + \frac{8}{x^2} y = 0$$

↑                      ↑

THE COEFFICIENT FUNCTIONS ARE NOT ANALYTIC AT  $x=0$ .

WE CANNOT USE A POWER SERIES AT  $x=0$ .

THE EQUATION IS A CAUCHY-EULER EQUATION.

THE SUBSTITUTION  $x = e^t$  TRANSFORMS IT TO LINEAR, CONST. COEFF.