Math 240 - Final Exam A

December 9, 2021

Name: key

Score ________

Show all work to receive full credit. You must work individually. This test is due December 14. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD. All integration must be done by hand, unless otherwise indicated.

---

1. (10 points) A tank initially contains 50 gal of pure water. A salt solution containing 2 lb of salt per gallon enters the tank at 3 gal/min and is uniformly mixed. The mixed solution leaves the tank at 2 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. Use technology to sketch the graph of $A(t)$ for $0 \le t \le 30$. Attach your graph.

$\frac{2\text{ lb}}{\text{gal}} \times \frac{3\text{ gal}}{\text{min}}$

$V(0) = 50$

$A(0) = 0$.

$\frac{A(t)\text{ lb}}{V(t)\text{ gal}} \times \frac{2\text{ gal}}{\text{min}}$

$\underbrace{3\text{ gal/min IN vs. } 2\text{ gal/min OUT}}$

$\text{Volume} = V(t) = t + 50$

Model...

$$\frac{dA}{dt} = 6 - \frac{2A}{t+50}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{2}{t+50}A = 6$$

$$\mu(t) = e^{\int \frac{2}{t+50}\,dt} = e^{2\ln|t+50|} = (t+50)^2$$

$$(t+50)^2 A(t) = \int 6(t+50)^2\,dt$$

$$(t+50)^2 A(t) = 2(t+50)^3 + C$$

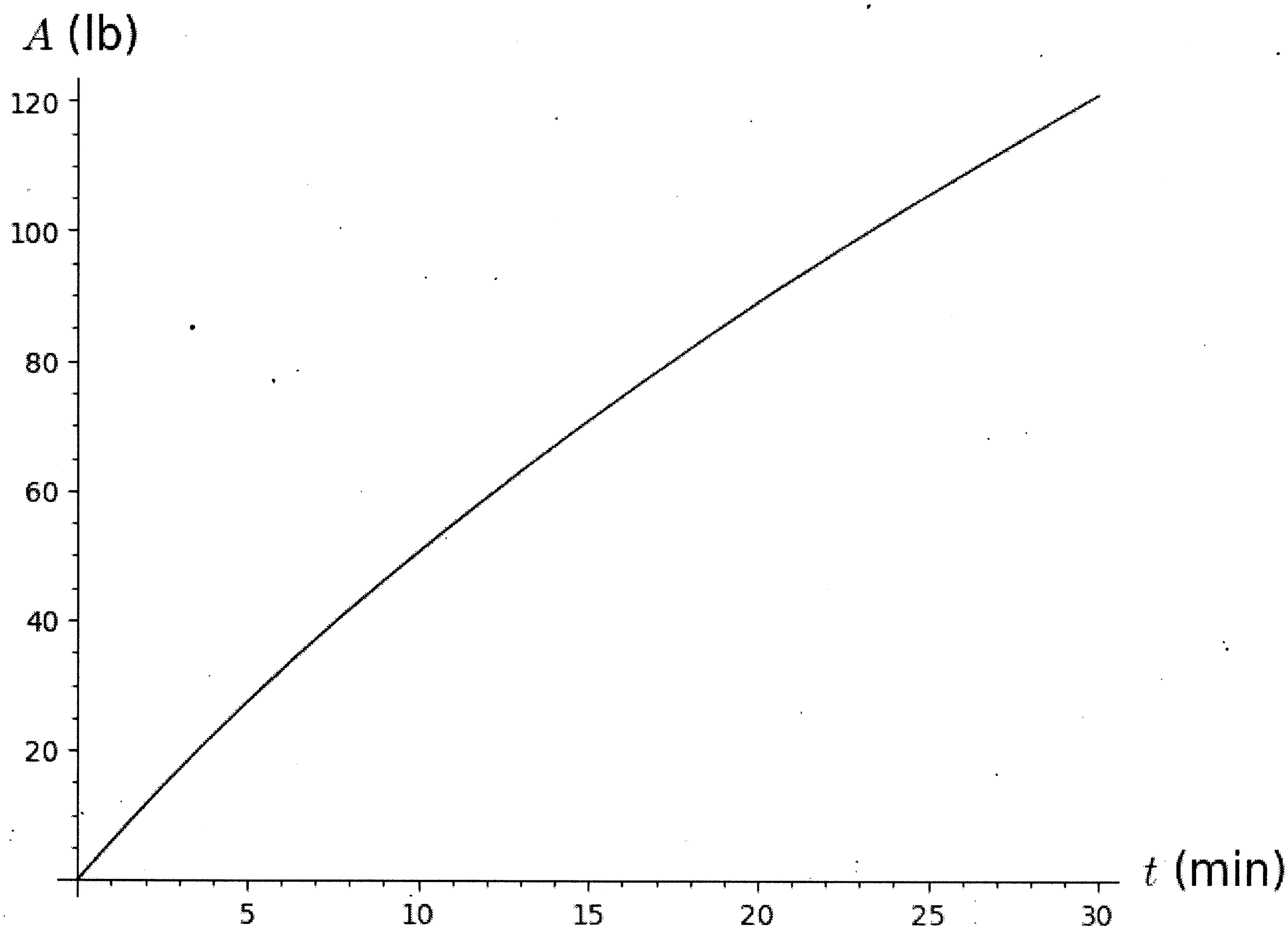
$$A(t) = 2(t+50) + \frac{C}{(t+50)^2}$$

$$A(0) = 0 \Rightarrow 100 + \frac{C}{2500} = 0$$

$$C = -250000$$

$$\boxed{A(t) = 2(t+50) - \frac{250000}{(t+50)^2}}$$

See attached graph.



A (lb)
120
100
80
60
40
20
5
10
15
20
25
30
t (min)

2. (16 points) The following differential equation falls into at least three of the named types of equations in section 1.6. Use two different approaches from section 1.6 to solve the equation. Use a word or short phrase to describe each approach.

$$3xy^2\frac{dy}{dx} + y^3 = x^3$$

(a) First approach:

EXACT ---

$$(y^3 - x^3)\,dx + 3xy^2\,dy = 0$$

$$\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x} = 3y^2$$

$$\frac{\partial F}{\partial x} = y^3 - x^3 \Rightarrow F(x,y) = xy^3 - \frac{1}{4}x^4 + g(y)$$

$$\frac{\partial F}{\partial y} = 3xy^2 \Rightarrow F(x,y) = xy^3 + h(x)$$

$$F(x,y) = xy^3 - \frac{1}{4}x^4 \Rightarrow$$

SOLUTION IS

$$xy^3 - \frac{1}{4}x^4 = C$$

OR $y = \sqrt[3]{\frac{C}{x} + \frac{1}{4}x^3}$

(b) Second approach:

$$3y^2\frac{dy}{dx} + \frac{1}{x}y^3 = x^2$$

BERNOULLI ---

$$u = y^3$$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{1}{x}u = x^2$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$$

$$|x|\,u(x) = \int |x|\,x^2\,dx$$

$$x\,u(x) = \int x^3\,dx$$

$$x\,u(x) = \frac{1}{4}x^4 + C$$

$$u(x) = \frac{1}{4}x^3 + \frac{C}{x}$$

$$y(x)^3 = \frac{1}{4}x^3 + \frac{C}{x}$$

$$y(x) = \sqrt[3]{\frac{1}{4}x^3 + \frac{C}{x}}$$

3. (14 points) Consider the equation $y' + y = e^x$. This semester, we have studied at least four different ways to solve this equation. Solve the equation using two different approaches. Use a word or short phrase to describe each approach.

(a) First approach:

LINEAR ---

$$\mu(x) = e^{\int 1\,dx} = e^x$$

$$e^x y(x) = \int e^x e^x\,dx = \int e^{2x}\,dx$$

$$e^x y(x) = \frac{1}{2} e^{2x} + C$$

$$y(x) = \frac{1}{2} e^x + \frac{C}{e^x}$$

$$\boxed{y(x) = \frac{1}{2} e^x + Ce^{-x}}$$

(b) Second approach:

LAPLACE TRANSFORM ---

$$sY(s) - y(0) + Y(s) = \frac{1}{s-1}$$

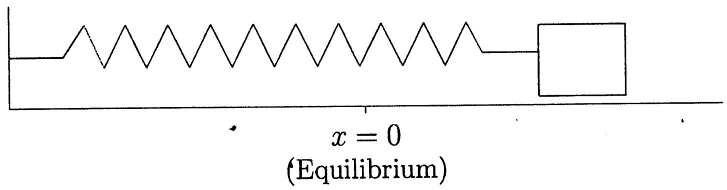
$$(s+1)Y(s) = \frac{1}{s-1} + y(0)$$

$$Y(s) = \frac{1}{s^2-1} + \frac{y(0)}{s+1}$$

$$y(x) = \frac{e^x - e^{-x}}{2} + y(0)e^{-x}$$

$$\boxed{y(x) = \frac{1}{2} e^x + Ce^{-x}, \text{ WHERE } C = y(0) - \frac{1}{2}}$$

4. (10 points) A 1-kg mass is attached to a spring with spring constant $3\,\mathrm{N/m}$. The damping constant for the system is $2\,\mathrm{N\text{-}sec/m}$. The mass is moved $0.5\,\mathrm{m}$ to the left of equilibrium (compressing the spring) and pushed to the right at $1\,\mathrm{m/s}$. Set up and solve the initial value problem that describes the motion. Write your final answer in terms of a single trigonometric function with phase shift.



$$x'' + 2x' + 3x = 0; \quad x(0) = -0.5, \quad x'(0) = 1$$

$$r^2 + 2r + 3 = 0$$

$$r^2 + 2r + 1 = -2$$

$$(r+1)^2 = -2$$

$$r = -1 \pm \sqrt{2}\,i$$

$$\alpha = -1, \quad \beta = \sqrt{2}$$

$$x(t) = c_1 e^{-t} \cos \sqrt{2}\,t + c_2 e^{-t} \sin \sqrt{2}\,t$$

$$x(0) = -0.5 \Rightarrow c_1 = -0.5$$

$$x'(t) = -c_1 e^{-t} \cos \sqrt{2}t - \sqrt{2}\, c_1 e^{-t} \sin \sqrt{2}t - c_2 e^{-t} \sin \sqrt{2}t + \sqrt{2}\, c_2 e^{-t} \cos \sqrt{2}t$$

$$x'(0) = 1 \Rightarrow -c_1 + \sqrt{2}\, c_2 = 1$$

$$\sqrt{2}\, c_2 = 0.5$$

$$c_2 = \frac{1}{2\sqrt{2}}$$

$$x(t) = -\frac{1}{2} e^{-t} \cos \sqrt{2}t + \frac{1}{2\sqrt{2}} e^{-t} \sin \sqrt{2}t$$

$$\underbrace{-\frac{1}{2} = A \sin \varphi, \quad \frac{1}{2\sqrt{2}} = A \cos \varphi}$$

$$\tan \varphi = -\sqrt{2} \quad \text{\& } \varphi \text{ IS IN QUAD 4}$$

$$\varphi = \tan^{-1}(-\sqrt{2})$$

OR

$$\varphi = -\tan^{-1}(\sqrt{2})$$

$$A = \sqrt{\frac{1}{4} + \frac{1}{8}} = \sqrt{\frac{3}{8}}$$

$$x(t) = \sqrt{\frac{3}{8}}\, e^{-t} \sin\left(\sqrt{2}t - \tan^{-1}(\sqrt{2})\right)$$

5. (10 points) Solve the following one-dimensional heat equation with Neumann boundary conditions. Rather than derive the solution method (as we did in class), you can use Theorem 2 on page 596. (You may use technology to evaluate the required integrals.)

$$5\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 10, \quad t > 0,$$

$$u_x(0,t) = u_x(10,t) = 0,$$

$$u(x,0) = 4x$$

$$\frac{\partial u}{\partial t} = \frac{1}{5}\frac{\partial^2 u}{\partial t^2} \; ; \; k = \frac{1}{5}, \; L = 10$$

SOLUTION IS $\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{10}x\right) e^{-n^2\pi^2 t/500}}$

WHERE $a_n = \frac{1}{5}\int_0^{10} 4x \cos\frac{n\pi}{10}x \, dx.$

$$a_0 = \frac{4}{5}\int_0^{10} x \, dx = \frac{2}{5}(10)^2 = 40 \Rightarrow \boxed{\frac{a_0}{2} = 20}$$

AND

$$\underbrace{a_n = 80\left[\frac{(-1)^n - 1}{\pi^2 n^2}\right]}, \quad n = 1, 2, 3, \ldots$$

$$\boxed{a_n = \begin{cases} \dfrac{-160}{\pi^2 n^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}}$$

Math 240 - Final Exam B

December 16, 2021

Name ____________________

Score __________

Show all work to receive full credit. All integration must be done by hand.

---

1. (10 points) Choose either ONE of following equations (A or B). Circle your choice and solve it. Ignore the other equation.

$$(A) \qquad (e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$$

$$(B) \qquad x^2y' + x(x+2)y = e^x$$

(A) $\frac{\partial}{\partial y}(e^{2y} - y\cos xy) = 2e^{2y} - \cos xy + xy\sin xy$

$\frac{\partial}{\partial x}(2xe^{2y} - x\cos xy + 2y) = 2e^{2y} - \cos xy + xy\sin xy$

Eq'n is EXACT.

$f_x(x,y) = e^{2y} - y\cos xy \Rightarrow f(x,y) = xe^{2y} - \sin xy + g(y)$

$f_y(x,y) = 2xe^{2y} - x\cos xy + 2y \Rightarrow f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$

$$f(x,y) = xe^{2y} - \sin xy + y^2$$

Solution is

$$xe^{2y} - \sin xy + y^2 = C.$$

(B) $\longrightarrow$

Turn over.

(B)

$$y' + \frac{x(x+2)}{x^2}\, y = \frac{e^x}{x^2}$$

OR

$$y' + \frac{x+2}{x}\, y = \frac{e^x}{x^2}$$

$$\mu(x) = e^{\int 1+\frac{2}{x}\,dx} = e^{x+2\ln|x|}$$

$$= x^2 e^x$$

$$x^2 e^x\, y(x) = \int x^2 e^x \left(\frac{e^x}{x^2}\right) dx$$

$$= \int e^{2x}\, dx$$

$$x^2 e^x\, y(x) = \frac{1}{2} e^{2x} + C$$

$$\boxed{y(x) = \frac{e^x}{2x^2} + \frac{C}{x^2 e^x}}$$

2. (10 points) Solve the differential equation.

$$y'' + 2y' = 2x + 5 - e^{-x}$$

Homo eqn : $y'' + 2y' = 0$

$r^2 + 2r = 0$

$r(r+2) = 0$

$r = 0,\ r = -2$

$y_h(x) = c_1 e^{0x} + c_2 e^{-2x}$

$= c_1 + c_2 e^{-2x}$

Non-Homo : $g_1(x) = 2x + 5$

$y_{p_1}(x) = x(Ax + B)$

$= Ax^2 + Bx$

Non-Homo : $g_2(x) = -e^{-x}$

$y_{p_2}(x) = Ce^{-x}$

$y_p(x) = Ax^2 + Bx + Ce^{-x}$

$y_p'(x) = 2Ax + B - Ce^{-x}$

$y_p''(x) = 2A + Ce^{-x}$

$y_p''(x) + 2y_p'(x) = 2A + Ce^{-x}$

$+ 4Ax + 2B - 2Ce^{-x}$

$4Ax + (2A + 2B) - Ce^{-x}$

$= 2x + 5 - e^{-x}$

$A = \frac{1}{2},\ B = 2,\ C = 1$

$y_p(x) = \frac{1}{2}x^2 + 2x + e^{-x}$

$$\boxed{y(x) = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + e^{-x}}$$

3. (10 points) State the recurrence relation that describes the coefficients of the power series solution centered at $x = 0$.

$$y'' - 2xy = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' - 2xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2a_n x^{n+1}.$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2a_{n-1} \right] x^n = 0$$

$$a_2 = 0 \quad \text{AND} \quad (n+2)(n+1) a_{n+2} = 2a_{n-1}; \quad n = 1, 2, 3, \ldots$$

a_0 AND a_1 ARE ARBITRARY.

$$a_2 = 0$$

$$a_{n+2} = \frac{2}{(n+2)(n+1)} a_{n-1}, \quad n = 1, 2, 3, \ldots$$

4. (10 points) Use Laplace transform techniques to solve the initial value problem.

$$y'' - 4y' + 4y = t^3 e^{2t}; \quad y(0) = 0,\ y'(0) = 0$$

$$\mathcal{L}\{y''\} - 4\,\mathcal{L}\{y'\} + 4\,\mathcal{L}\{y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$s^2 Y(s) - \cancel{s y(0)}^{0} - \cancel{y'(0)}^{0} - 4\left(s Y(s) - \cancel{y(0)}^{0}\right) + 4\,Y(s) = \frac{6}{(s-2)^4}$$

$$(s^2 - 4s + 4)\,Y(s) = \frac{6}{(s-2)^4}$$

$$(s-2)^2\,Y(s) = \frac{6}{(s-2)^4}$$

$$Y(s) = \frac{6}{(s-2)^6} = \frac{6}{5!}\,\frac{5!}{(s-2)^6}$$

$$y(t) = \frac{6}{5!}\,t^5 e^{2t}$$

$$y(t) = \frac{1}{20}\,t^5 e^{2t}$$