Math 240 - Quiz 4

September 21, 2023

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. This quiz is due September 26.

1. (2.5 points) We will soon learn a very easy way to solve the equation $y'' + y' = 0$. For now, solve by using an appropriate substitution to reduce the equation to 1st-order.

Type I:

$u = y'$, $u' = y''$ $\Rightarrow$ $u' + u = 0$

$\Rightarrow \dfrac{du}{dx} = -u \Rightarrow \dfrac{1}{u}\,dx = -dx$

$\ln|u| = -x + C$ $\quad u = ke^{-x}$

$y' = ke^{-x}$

$y = -ke^{-x} + c$

OR

$\boxed{y = c_1 + c_2 e^{-x}}$

2. (2.5 points) Solve: $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{3xy}$

$\dfrac{dy}{dx} = \dfrac{x}{3y} - \dfrac{y}{3x}$

LET $u = \dfrac{y}{x}$ SO THAT

$y = ux$ & $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$u + x\dfrac{du}{dx} = \dfrac{1}{3u} - \dfrac{u}{3}$

$x\dfrac{du}{dx} = \dfrac{1}{3u} - \dfrac{4u}{3}$

$x\dfrac{du}{dx} = \dfrac{1-4u^2}{3u}$

$w = 1-4u^2$ SUBS

$\dfrac{3u}{1-4u^2}\,du = \dfrac{1}{x}\,dx$

$-\dfrac{3}{8}\ln|1-4u^2| = \ln|x| + C_1$

$\ln|1-4u^2| = \ln|x|^{-8/3} + C_2$

$1-4u^2 = C_3 x^{-8/3}$

$4u^2 = 1 - C_3 x^{-8/3}$

$u^2 = \dfrac{1}{4} - C_4 x^{-8/3}$

$\boxed{y^2(x) = \dfrac{1}{4}x^2 - Cx^{-2/3}}$

Turn over.

3. (2.5 points) Solve: $\dfrac{dx}{dt} + tx^3 + \dfrac{x}{t} = 0$

$\boxed{X(t) \equiv 0 \text{ IS ALSO A SOLUTION}}$

$$\frac{dx}{dt} + \frac{1}{t}x = -tx^3 \Rightarrow x^{-3}\frac{dx}{dt} + \frac{1}{t}x^{-2} = -t$$

$$u = x^{-2}$$

$$\frac{du}{dt} = -2x^{-3}\frac{dx}{dt}$$

$$-\frac{1}{2}\frac{du}{dt} + \frac{1}{t}u = -t$$

$$\frac{du}{dt} - \frac{2}{t}u = 2t$$

$$\mu(t) = e^{-\int \frac{2}{t}dt} = e^{-2\ln|t|} = \frac{1}{t^2}$$

$$\frac{1}{t^2}u = \int \frac{2}{t}dt = 2\ln|t| + C$$

$$u = 2t^2\ln|t| + Ct^2$$

$$\frac{1}{x^2} = u \Rightarrow \boxed{X(t) = \frac{1}{\sqrt{2t^2\ln|t| + Ct^2}}}$$

4. (2.5 points) Solve by using an appropriate substitution to reduce the equation to 1st-order.

$$xy'' - y' = 3x^2$$

Type I:

$u = y'$
$u' = y''$

$$xu' - u = 3x^2$$

$$u' - \frac{1}{x}u = 3x$$

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|} = \frac{1}{x},\quad x > 0$$

$$\frac{1}{x}u = \int 3\,dx = 3x + C$$

$$u = 3x^2 + Cx$$

$$\boxed{y(x) = x^3 + Cx^2 + D}$$