Math 240 - Test 1

September 14, 2023

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (5 points) Make up an example of an initial value problem for a 3rd-order, ordinary, linear differential equation.

$$5y''' + 2y'' - 3y' + 10y = 5\sin 2x \,;$$
$$y(0) = 1,\; y'(0) = -1,\; y''(0) = 0$$

2. (3 points) Make up an example of a 2nd-order, partial diffential equation with dependent variable u and independent variables x and y.

$$\frac{\partial u}{\partial x} = 5u + \frac{\partial^2 u}{\partial y^2}$$

3. (3 points) Which of the following equations are linear? Select all that apply.

(a) $\dfrac{dx}{dt} = (4 - x)(1 - x)$

(b) $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 9y = 2\cos 3x$

(c) $\sqrt{1-x}\,\dfrac{d^2x}{dt^2} + 2t\dfrac{dx}{dt} = 0$

(d) $\dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial z}{\partial x} + 2z$

(e) $x\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + xy = 0$

4. (6 points) Use Euler's method with $h = 0.1$ to approximate the value of $y(2.3)$ for the initial value problem $y' = x\sqrt{y}$, $y(2) = 9$.

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_0 = 9$
$x_0 = 2$

$y_1 = 9 + 0.1(2)\sqrt{9}$
$= 9.6 \Rightarrow y(2.1) \approx 9.6$

$x_1 = 2.1$

$y_2 = 9.6 + 0.1(2.1)\sqrt{9.6}$
$= 10.250661 \Rightarrow y(2.2) \approx 10.25$

$x_2 = 2.2$

$y_3 = y_2 + 0.1(2.2)\sqrt{y_2}$
$= 10.955028$

$x_3 = 2.3$

$y(2.3) \approx 10.96$

5. (8 points) Consider the initial value problem:

$$\frac{dy}{dx} = x\sqrt{y} \quad y(2) = 9.$$

(a) Solve the initial value problem.

$y^{-1/2}\,dy = x\,dx$

$\int y^{-1/2}\,dy = \int x\,dx$

$2y^{1/2} = \frac{1}{2}x^2 + C$

$y(2) = 9 \Rightarrow 2(3) = \frac{1}{2}(4) + C \Rightarrow C = 4$

$y^{1/2} = \frac{1}{4}x^2 + 2$

$y(x) = \left(\frac{1}{4}x^2 + 2\right)^2$

(b) Compute $y(2.3)$ and compare your answer to your result in problem 4.

$y(2.3) = \left(\frac{5.29}{4} + 2\right)^2 \approx 11.04$

EULER'S METHOD GAVE ABOUT ONE CORRECT DECIMAL DIGIT.

6. (12 points) Analyze each initial value problem to determine which one of these applies.

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $y' - 3y^{2/3} = 0, \quad y(2) = 0$

$f(x,y) = 3y^{2/3}$ ← CONTINUOUS EVERYWHERE IN $\mathbb{R}^2$

$f_y(x,y) = 2y^{-1/3}$ ← NOT CONT. WHEN $y = 0$

$\boxed{(A)}$

(b) $3x\frac{dy}{dx} + x^2y = 1, \quad y(1) = -2$

$f(x,y) = \frac{1 - x^2y}{3x}, \quad f_y(x,y) = \frac{-x^2}{3x}$

THESE ARE BOTH CONTINUOUS AS LONG AS WE STAY AWAY FROM $x = 0$.

$\boxed{(B)}$

(c) $\left(\frac{dy}{dx}\right)^2 + y^2 + 3 = 0, \quad y(0) = 0$

$f(x,y) = \sqrt{-y^2 - 3}$ ← THIS IS NOT DEFINED FOR ANY y.

$\boxed{(C)}$

IN FACT, A REAL SOLUTION CANNOT EXIST.

7. (12 points) Solve: $t^3\dfrac{dx}{dt} + 3t^2x = t, \quad x(2) = 0$

$$x' + \frac{3}{t}x = \frac{1}{t^2} \qquad p(t) = \frac{3}{t}, \quad q(t) = \frac{1}{t^2}$$

$$\mu(t) = e^{\int \frac{3}{t}\,dt}$$

$$= e^{3\ln|t|} = |t|^3 = t^3, \quad t > 0 \leftarrow$$ THIS ASSUMPTION IS REASONABLE GIVEN THE INITIAL CONDITION.

$$t^3 x = \int t^3\left(\frac{1}{t^2}\right)dt = \int t\,dt$$

$$= \frac{1}{2}t^2 + C$$

$$x = \frac{1}{2t} + \frac{C}{t^3}$$

$$\boxed{x(t) = \frac{1}{2t} - \frac{2}{t^3}}$$

$$x(2) = 0 \Rightarrow \frac{1}{4} + \frac{C}{8} = 0 \Rightarrow C = -2$$

8. (7 points) Solve: $\dfrac{dy}{dx} = x\sqrt{x^2+9}, \quad y(-4) = 0$

$$y(x) = \int x\sqrt{x^2+9}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2+9)^{3/2} + C$$

$u = x^2 + 9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$y(-4) = 0 \Rightarrow$$

$$\frac{1}{3}(25)^{3/2} + C = 0$$

$$C = -\frac{25^{3/2}}{3} = -\frac{125}{3}$$

$$\boxed{y(x) = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}}$$

9. (2 points) At the start of an experiment, a motorboat is coasting at 20 mph. The time rate of change of the boat's velocity is proportional to the square of its velcoity. Write the corresponding initial value problem.

$$\boxed{\frac{dv}{dt} = kv^2, \quad V(0) = 20}$$

10. (12 points) Consider the following initial value problem:

$$\underbrace{(1/x + 2y^2x)}_{M(x,y)}\, dx + \underbrace{(2yx^2 - \cos y)}_{N(x,y)}\, dy = 0, \quad y(1) = \pi.$$

(a) Use the test for exactness to show that the equation is exact?

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 + 4yx = 4yx \\ \frac{\partial N}{\partial x} &= 4yx - 0 = 4yx \end{aligned}\right\}$$

SAME! EQUATION IS EXACT.

(b) Solve the initial value problem.

$$F_x(x,y) = \frac{1}{x} + 2y^2x \Rightarrow F(x,y) = \ln|x| + y^2x^2 + g(y)$$

$$F_y(x,y) = 2yx^2 - \cos y \Rightarrow F(x,y) = y^2x^2 - \sin y + h(x)$$

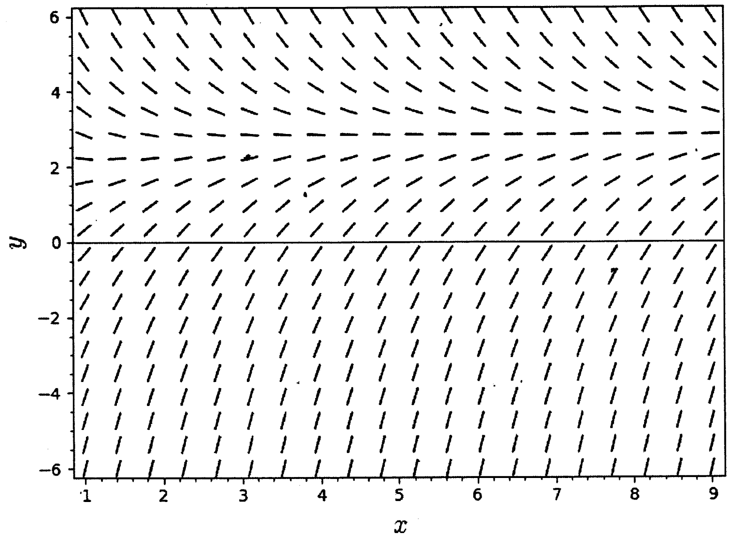
$$F(x,y) = \ln|x| - \sin y + y^2x^2 = C$$

$$F(1,\pi) = 0 - 0 + \pi^2 = C$$

$$\boxed{\ln x - \sin y + y^2x^2 = \pi^2, \quad x > 0}$$

(c) Is your solution explicit or implicit?

11. (10 points) Consider the differential equation $\frac{dy}{dx} = 3 - y + \frac{1}{x}$. A portion of its slope field is shown below.



(a) What is the slope of the solution curve passing through (5, 10)?

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(5,10)} = 3 - 10 + \frac{1}{5} = \boxed{-\frac{34}{5}}$$

(b) Suppose you are given some initial condition $y(x_0) = y_0$, where $x_0 > 0$. Make a conjecture about the values of $y(x)$ as $x \to \infty$.

IT LOOKS LIKE $\boxed{y(x) \to 3}$ AS $X \to \infty$.

(c) The general solution of the equation involves a non-elementary function known as the *exponential integral*. Carry out the solution process as far as you can.

$$\frac{dy}{dx} + y = 3 + \frac{1}{x}$$

$$p(x) = 1 \quad q(x) = 3 + \frac{1}{x}$$

$$\mu(x) = e^{\int dx} = e^x$$

$$e^x y = \int \left(3 + \frac{1}{x}\right) e^x \, dx = \int \left(3e^x + \frac{e^x}{x}\right) dx$$

$$= 3e^x + \int \frac{e^x}{x} dx$$

$$\boxed{y(x) = 3 + \frac{1}{e^x} \int \frac{e^x}{x} dx} = 3 + \frac{\text{Eint}(x) + C}{e^x}$$

(d) (2 points extra credit) Your general solution in part (c) should support your conjecture in part (b). Show that it does.

NEED TO SHOW $\frac{\int \frac{e^x}{x} dx}{e^x} \to 0$ AS $X \to \infty$.

USE L'HOPITAL'S RULE...

$$\lim_{x \to 0} \frac{\int \frac{e^x}{x} dx}{e^x} = \lim_{x \to 0} \frac{\frac{e^x}{x}}{e^x} = \lim_{x \to 0} \frac{1}{x} = 0$$

The following problems make up the take-home portion of the test. These problems are due September 19, 2023. You must work on your own.

12. (5 points) The following equation is called a *homogeneous equation*. It is not separable, but it can be transformed to a separable equation by an appropriate substitution. See pages 60–61 of our textbook. Then use the appropriate substitution to solve this equation.

$$xy^2y' = x^3 + y^3$$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} = \frac{x^2}{y^2} + \frac{y}{x}$$

LET $u = \frac{y}{x}$ SO THAT $y = ux$

AND $\frac{dy}{dx} = u + x\frac{du}{dx}$

$$\cancel{u} + x\frac{du}{dx} = \frac{1}{u^2} + \cancel{u}$$

$$\frac{du}{dx} = \frac{1}{xu^2} \quad \leftarrow \text{SEPARABLE!}$$

$$u^2\,du = \frac{1}{x}\,dx$$

$$\frac{1}{3}u^3 = \ln|x| + C_0$$

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C_1$$

$$y^3 = 3x^3\ln|x| + C_1x^3$$

$$y(x) = \sqrt[3]{3x^3\ln|x| + C_1x^3}$$

- OR -

$$\boxed{y(x) = x\sqrt[3]{3\ln|x| + C}}$$

13. (7 points) Suppose T is the temperature of an object at time t. Newton's law of cooling states that $dT/dt = k(T - T_s)$, where T_s is the surrounding (constant) temmperature.

(a) Find the general solution of the differential equation.

$$\frac{1}{T-T_s}\,dT = k\,dt$$

$$\ln|T-T_s| = kt + C_0$$

$$|T-T_s| = e^{kt+C_0} = C_1 e^{kt}$$

$$T - T_s = C_2 e^{kt}$$

$$\boxed{T(t) = T_s + Ce^{kt}}$$

(b) A pot of boiling water at 100° C is removed from a stove and left to cool. After 5 min, the water temperature is 80° C, and after another 5 min, it has dropped to 65° C. Assuming Newton's law of cooling, find the surrounding temperature.

$$T(0) = 100 = T_s + C \Rightarrow T_s = 100 - C$$

$$\left.\begin{aligned} T(5) &= 80 = T_s + Ce^{5k} \\ T(10) &= 65 = T_s + Ce^{10k} \end{aligned}\right\} \quad \begin{aligned} -20 &= -C + Ce^{5k} \\ -35 &= -C + Ce^{10k} \end{aligned} \Rightarrow \frac{4}{7} = \frac{-1+e^{5k}}{-1+e^{10k}}$$

$$\Downarrow$$

$$-4 + 4e^{10k} = -7 + 7e^{5k}$$

$$4e^{10k} - 7e^{5k} + 3 = 0$$

$$(4e^{5k} - 3)(e^{5k} - 1) = 0$$

$$\boxed{e^{5k} = \frac{3}{4}} \quad \text{OR} \quad e^{5k} = 1$$

$$5k = \ln(0.75) \qquad\qquad k = 0$$

$$k = \frac{\ln(0.75)}{5} \qquad\qquad \text{(Nope!)}$$

(THIS WOULD MAKE T CONST.)

$$-20 = -C + \frac{3}{4}C$$

$$-20 = -\frac{1}{4}C$$

$$C = 80$$

$$\Downarrow$$

$$\boxed{T_s = 20°C}$$

14. (8 points) A 1000-L tank initially contains 500 L of a salt solution in which 5 kg of salt are dissolved. Brine (salt water) with a salt concentration of 0.2 kg/L enters the tank at a rate of 5 L/min. The liquid is kept uniform by stirring and flows out of the tank at 4 L/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. How much salt is in the tank when it is full?

$$0.2\,\frac{kg}{L} \times \frac{5L}{min} = 1\ kg/min$$

$$\boxed{\begin{array}{l} A(0) = 5 \\ V(0) = 500 \end{array}}$$

VOLUME OF TANK INCREASES AT 1 L/min.

$$V(t) = 500 + t, \quad 0 \le t \le 500$$

$$\frac{4L}{min} \cdot \frac{A(t)\ kg}{(500+t)\ L} = \frac{4A}{500+t}\ kg/min$$

$$\frac{dA}{dt} = 1 - \frac{4A}{500+t}; \quad A(0) = 5$$

$$\frac{dA}{dt} + \frac{4}{500+t}A = 1$$

$$\mu(t) = e^{\int \frac{4}{500+t}\,dt} = e^{4\ln|500+t|} = (500+t)^4$$

$$(500+t)^4 A = \int (500+t)^4\,dt = \frac{1}{5}(500+t)^5 + C$$

$$A = \frac{1}{5}(500+t) + \frac{C}{(500+t)^4}$$

$$A(0) = 5 \Rightarrow 100 + \frac{C}{500^4} = 5$$

$$C = -95(500)^4$$

$$A(t) = \frac{1}{5}(500+t) - \frac{95(500)^4}{(500+t)^4}$$

$$A(500) = \frac{1000}{5} - \frac{95(500)^4}{(1000)^4} = 200 - \frac{95}{16} = \boxed{194.0625\ kg}$$

THAT'S LOTS OF SALT!