Math 240 - Test 2
October 12, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (10 points) Assume $x > 0$ and solve the following initial value problem.

$$y' + \frac{4}{x}y = x^3y^2, \quad y(1) = 1/2$$

BERNOULLI EQUATION

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

$$u = y^{-1}$$

$$\frac{du}{dx} = -1y^{-2}\frac{dy}{dx}$$

$$-\frac{du}{dx} + \frac{4}{x}u = x^3$$

$$\frac{du}{dx} - \frac{4}{x}u = -x^3$$

$$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = \frac{1}{x^4}$$

$$\frac{1}{x^4}u(x) = \int -\frac{1}{x}dx$$

$$\frac{1}{x^4}u(x) = -\ln x + C, \quad x > 0$$

$$u(x) = -x^4\ln x + Cx^4$$

$$y(x) = \frac{1}{Cx^4 - x^4\ln x}$$

$$y(1) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{C} \Rightarrow C = 2$$

$$\boxed{y(x) = \frac{1}{2x^4 - x^4\ln x}}$$

2. (4 points) Explain why the functions $y_1(x) = x^2+1$, $y_2(x) = x^2+3x$, and $y_3(x) = 1-3x$ are linearly **dependent**.

y_1 IS A LINEAR COMBINATION OF y_2 & y_3.

IN PARTICULAR, $y_1(x) = y_2(x) + y_3(x)$.

3. (8 points) Find two linearly independent solutions of $y'' + 3y' - 18y = 0$. Use the Wronskian to show that your solutions are indeed independent.

CHAR. EQN: $r^2 + 3r - 18 = 0$

$(r+6)(r-3) = 0$

$r = -6, \ r = 3$

$$\boxed{\begin{aligned} y_1(x) &= e^{-6x} \\ y_2(x) &= e^{3x} \end{aligned}}$$

$$W = \begin{vmatrix} e^{-6x} & e^{3x} \\ -6e^{-6x} & 3e^{3x} \end{vmatrix} = 3e^{-3x} + 6e^{-3x}$$

$$= 9e^{-3x}$$

AND THIS IS NEVER EQUAL TO ZERO.

$\Downarrow$

y_1 & y_2 ARE LIN. INDEP.

4. (10 points) Solve the initial value problem.

$$y''' + 2y' = 0; \quad y(0) = 1, \, y'(0) = -1, \, y''(0) = 4$$

CHAR. EQN:

$r^3 + 2r = 0$

$r(r^2+2) = 0$

$r = 0, \ r = \pm\sqrt{2}\, i$

$y(x) = c_1 e^{0x} + c_2 \cos\sqrt{2}\,x + c_3 \sin\sqrt{2}\,x$

$y(x) = c_1 + c_2 \cos\sqrt{2}\,x + c_3 \sin\sqrt{2}\,x$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(x) = -\sqrt{2}\, c_2 \sin\sqrt{2}\,x + \sqrt{2}\, c_3 \cos\sqrt{2}\,x$

$y''(x) = -2c_2 \cos\sqrt{2}\,x - 2c_3 \sin\sqrt{2}\,x$

$y'(0) = -1 \Rightarrow \sqrt{2}\, c_3 = -1 \Rightarrow c_3 = -\frac{1}{\sqrt{2}}$

$y''(0) = 4 \Rightarrow -2c_2 = 4 \Rightarrow c_2 = -2$

$c_1 = 3$

$y(x) = 3 - 2\cos\sqrt{2}\,x - \frac{1}{\sqrt{2}} \sin\sqrt{2}\,x.$

5. (12 points) Consider the equation $(x^2 - 1)y'' - 6xy' + 12y = 0$.

(a) The functions $y_1(x) = 1 + 6x^2 + x^4$ and $y_2(x) = x + x^3$ are solutions. Choose either one of them and verify that it is a solution.

$y_2(x) = x + x^3$

$y_2'(x) = 1 + 3x^2$ $\rightarrow$ $(x^2-1)(6x) - 6x(1+3x^2) + 12(x+x^3)$

$y_2''(x) = 6x$

$= 6x^3 - 6x - 6x + 18x^3 + 12x + 12x^3 = 0$ ✓

(b) Find another solution (other than $y(x) \equiv 0$).

ANY NONZERO LINEAR COMBO OF y_1 & y_2 WILL WORK, e.g.,

$$\boxed{y(x) = 1 + 6x^2 + x^4 + x + x^3}$$

(c) Without using the Wronskian, argue that y_1 and y_2 are linearly independent.

y_1 IS NOT A MULTIPLE OF y_2.

(d) Should we expect a unique solution satisfying $y(1) = 2, y'(1) = 5$? Explain your reasoning.

$$y'' - \frac{6x}{x^2-1}y' - \frac{12}{x^2-1}y = 0$$

NO, THE COEFFICIENT FUNCTIONS ARE NOT CONT. AT $x = 1$. OUR THEOREM DOES NOT APPLY.

(e) Use what you've learned in parts (a) and (c) to find the solution of the IVP $(x^2 - 1)y'' - 6xy' + 12y = 0$; $y(0) = 2$, $y'(0) = 5$.

$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1(1 + 6x^2 + x^4) + c_2(x + x^3)$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(x) = c_1(12x + 4x^3) + c_2(1 + 3x^2)$

$y'(0) = 5 \Rightarrow c_2 = 5$

$$\boxed{y(x) = 2(1 + 6x^2 + x^4) + 5(x + x^3)}$$

6. (10 points) Assuming $x > 0$, find the general solution of the Cauchy-Euler equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = 0$.

$X = e^t$ TRANSFORMS EQUATION IT

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 0$$

CHAR. EQN:

$$r^2 - 4r + 5 = 0$$

$$r^2 - 4r + 4 = -1$$

$$(r-2)^2 = -1$$

$$r = 2 \pm i$$

$$\alpha = 2, \quad \beta = 1$$

$$y(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$$

$$X = e^t, \quad \ln x = t$$

$$\Downarrow$$

$$y(x) = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x)$$

7. (6 points) The general solution of a homogeneous, constant-coefficient, linear differential equation is $y(x) = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$. Find such an equation.

CHAR. EQN MUST BE $(r-2)^3 = (r-2)(r-2)(r-2) = (r-2)(r^2-4r+4)$

$$= r^3 - 6r^2 + 12r - 8$$

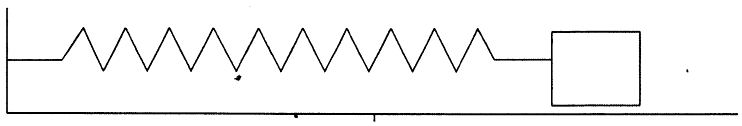
$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$

8. (4 points) A mass-spring system is described by the equation $mx'' + bx' + kx = 0$. What must be true if the system is underdamped? Describe the motion of the mass in such a case.

$b \neq 0, \quad b^2 - 4mk < 0$

OSCILLATIONS WILL OCCUR BUT THEY WILL DECAY OVER TIME.

9. (16 points) A 4-kg mass is attached to a spring with spring constant 17 N/m. The damping constant for the system is 4 N-sec/m. The mass is moved 3 m to the **right** of equilibrium (stretching the spring) and pushed to the **left** with a speed of 9.5 m/sec. Find the equation of motion. Write your solution in terms of a single sine or cosine with a phase shift.



$x = 0$
(Equilibrium)

$$4x'' + 4x' + 17x = 0$$

$$x(0) = 3, \quad x'(0) = -9.5$$

CHAR EQN:

$$4r^2 + 4r + 17 = 0$$

$$r^2 + r + \frac{17}{4} = 0$$

$$\left(r + \frac{1}{2}\right)^2 + 4 = 0$$

$$r = -\frac{1}{2} \pm 2i$$

$$x(t) = c_1 e^{-t/2} \cos 2t + c_2 e^{-t/2} \sin 2t$$

$$x(0) = 3 \Rightarrow c_1 = 3$$

$$x'(0) = -9.5 \Rightarrow -\frac{1}{2}c_1 + 2c_2 = -9.5$$

$$2c_2 = -8$$

$$c_2 = -4$$

$$x(t) = 3e^{-t/2} \cos 2t - 4e^{-t/2} \sin 2t$$

$$= Ae^{-t/2} \sin(2t + \varphi)$$

$$A = \sqrt{9 + 16} = 5$$

$$c_1 = 5 \sin \varphi = 3$$
$$c_2 = 5 \cos \varphi = -4$$

$\Rightarrow \varphi$ IS IN QUAD 2

AND

$$\tan \varphi = -\frac{3}{4}$$

$$\boxed{x(t) = 5e^{-t/2} \sin\left(2t + \tan^{-1}\left(-\frac{3}{4}\right) + \pi\right)}$$

The following problems make up the take-home portion of the test. These problems are due October 17, 2023. You must work on your own.

10. (5 points) Solve: $x^2y' = xy + x^2e^{y/x}$

$$y' = \frac{y}{x} + e^{y/x}$$

$$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = u + e^{u}$$

$$x\frac{du}{dx} = e^{u} \Rightarrow e^{-u}\,du = \frac{1}{x}\,dx$$

$$-e^{-u} = \ln|x| + C_1$$

$$e^{-u} = C_2 - \ln|x|$$

$$-u = \ln(C_2 - \ln|x|)$$

$$\frac{y}{x} = -\ln(C_2 - \ln|x|)$$

$$\boxed{y(x) = -x\ln(C - \ln|x|)}$$

11. (7 points) Find the general solution. You may use a computer algebra system to factor the characteristic polynomial.

$$y^{(7)} - 3y^{(6)} + 2y^{(5)} - 6y^{(4)} + y''' - 3y'' = 0$$

$$r^7 - 3r^6 + 2r^5 - 6r^4 + r^3 - 3r^2 = 0$$

$$r^2(r-3)(r^2+1)^2 = 0$$

$$r = 0,\ r = 0,\ r = 3,\ r = \pm i,\ r = \pm i$$

$$y_1 = e^{0x} = 1$$

$$y_2 = xe^{0x} = x$$

$$y_3 = e^{3x}$$

$$y_4 = e^{0x}\cos x = \cos x$$

$$y_5 = e^{0x}\sin x = \sin x$$

$$y_6 = xe^{0x}\cos x = x\cos x$$

$$y_7 = xe^{0x}\sin x = x\sin x$$

$$y(x) = C_1 + C_2x + C_3e^{3x} + C_4\cos x + C_5\sin x + C_6x\cos x + C_7x\sin x$$

12. (8 points) Solve: $y^3 y'' = 1$

Let $u = y'$, $y'' = u\frac{du}{dy}$

$$y^3 u \frac{du}{dy} = 1 \Rightarrow u\,du = y^{-3}\,dy$$

$$\frac{1}{2}u^2 = -\frac{1}{2}y^{-2} + C_1$$

$$u^2 = C_2 - y^{-2}$$

$$u^2 = \frac{C_2 y^2 - 1}{y^2}$$

$$y' = \sqrt{\frac{C_2 y^2 - 1}{y^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{C_2 y^2 - 1}}{y}, \quad y > 0$$

$$\frac{y}{\sqrt{C_2 y^2 - 1}}\,dy = dx$$

$$\int \frac{y}{\sqrt{C_2 y^2 - 1}}\,dy = \int 1\,dx = x + d$$

$w = C_2 y^2 - 1$

$dw = 2C_2 y\,dy \qquad \int \frac{1}{2C_2} w^{-1/2}\,dw$

$$\frac{1}{C_2} w^{1/2} = x + d$$

$$\frac{1}{C_2}\sqrt{C_2 y^2 - 1} = x + d$$

$$C_2 y^2 - 1 = (C_2 x + C_3)^2$$

$$C_2 y^2 = (C_2 x + C_3)^2 + 1 = C_2^2 x^2 + 2C_2 C_3 x + C_3^2 + 1$$

$$y^2 = C_2 x^2 + 2C_3 x + \frac{C_3^2}{C_2} + \frac{1}{C_2}$$

$$y(x) = \sqrt{C_2 x^2 + 2C_3 x + \frac{C_3^2}{C_2} + \frac{1}{C_2}}, \quad y > 0$$