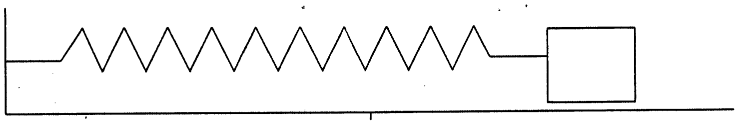
Math 240 - Test 3
November 9, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

1. (8 points) A 2-kg mass is attached to a spring with spring constant 18 N/m. The damping constant for the system is 12 N-sec/m. The mass is moved 2 m to the **right** of equilibrium (stretching the spring) and pushed to the **left** with a speed of 1 m/sec. State whether the system is overdamped, underdamped, or critically damped; and find the equation of motion.



$x = 0$
(Equilibrium)

$2x'' + 12x' + 18x = 0$

$x(0) = 2, \quad x'(0) = -1$

$2r^2 + 12r + 18 = 0$

$2(r^2 + 6r + 9)$

$= 2(r+3)^2 = 0$

$r = -3, \; r = -3 \Rightarrow$ $\boxed{\text{CRITICALLY DAMPED}}$

$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$x(0) = 2 \Rightarrow c_1 = 2$

$x'(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$

$x'(0) = -1 \Rightarrow -3c_1 + c_2 = -1$

$-6 + c_2 = -1$

$c_2 = 5$

$x(t) = 2e^{-3t} + 5te^{-3t}$

2. (8 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes one of the following situations: *simple harmonic motion, underdamped motion, overdamped motion,* or *critically damped motion.* Tell which situation coincides with each equation.

(a) $x(t) = \underbrace{\sqrt{6}e^{-t/2}\sin\left(t+\frac{\pi}{3}\right)}_{\text{DAMPED OSCILLATIONS}}$ $\boxed{\text{UNDERDAMPED}}$

(b) $2x'' + 4x' + 3x = 0$

$b^2 - 4mk = 16 - 4(2)(3) = 16 - 24 < 0 \Rightarrow \boxed{\text{UNDERDAMPED}}$

(c) $5x'' + 17x = 0$

$b = 0 \Rightarrow \boxed{\text{SIMPLE HARMONIC MOTION}}$

(d) $x(t) = 5e^{-2t} - 3e^{-4t}$

$r = -2,\ r = -4$ $\boxed{\text{OVERDAMPED}}$

3. (10 points) Use variation of parameters to solve the following differential equation.

$$y'' + y = \csc^2 x$$

Homo. EQN: $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_c(x) = C_1 \cos x + C_2 \sin x$

NON HOMO HAS $g(x) = \csc^2 x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$V_2(x) = \int \csc^2 x \cos x \, dx = \int \cot x \csc x \, dx = -\csc x$$

$$V_1(x) = \int -\csc^2 x \sin x \, dx = -\int \csc x \, dx = \ln|\csc x + \cot x|$$

$$y_p(x) = \underbrace{-\csc x \sin x}_{-1} + \cos x \ln|\csc x + \cot x|$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x - 1 + \cos x \ln|\csc x + \cot x|}$$

4. (12 points) Find the general solution: $y^{(3)} + y'' = 3e^x + 4x^2$

Homo eqn: $y''' + y'' = 0$

$r^3 + r^2 = 0 \quad r^2(r+1) = 0$

$r = 0, \ r = 0, \ r = -1$

$y_c(x) = c_1 + c_2 x + c_3 e^{-x}$

Non homo has $g(x) = 3e^x + 4x^2$

$y_p(x) = x^{s_1} A e^x + x^{s_2}\left(Bx^2 + Cx + D\right)$

$s_1 = 0 \ , \quad s_2 = 2$

$y_p(x) = Ae^x + Bx^4 + Cx^3 + Dx^2$

$y_p'(x) = Ae^x + 4Bx^3 + 3Cx^2 + 2Dx$

$y_p''(x) = Ae^x + 12Bx^2 + 6Cx + 2D$

$y_p'''(x) = Ae^x + 24Bx + 6C$

$y_p''' + y_p'' = 2Ae^x + 12Bx^2 + (6C + 24B)x + (2D + 6C)$

$= 3e^x + 4x^2$

$\Rightarrow \quad 2A = 3 \qquad A = \frac{3}{2}$

$12B = 4 \qquad B = \frac{1}{3}$

$6C + 24B = 0 \qquad C = -\frac{4}{3}$

$2D + 6C = 0 \qquad D = 4$

$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2$

5. (8 points) Consider the following equation:

$$y'' - 10y' + 25y = 5x^2e^{5x}.$$

Solve the corresponding homogeneous equation. Then use your table to find the appropriate form of the particular solution for the nonhomogeneous equation. **Do not solve for the undetermined coefficients.**

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r = 5,\ r = 5$

$\boxed{y_c(x) = c_1 e^{5x} + c_2 x e^{5x}} \longrightarrow$

$g(x) = 5x^2 e^{5x}$

$y_p(x) = x^s\left(Ax^2 + Bx + C\right)e^{5x}$

$s = 2$

$\boxed{y_p(x) = \left(Ax^4 + Bx^3 + Cx^2\right)e^{5x}}$

6. (6 points) For each equation below, consider a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Determine the minimum radius of convergence that is guaranteed by the theorem we discussed in class.

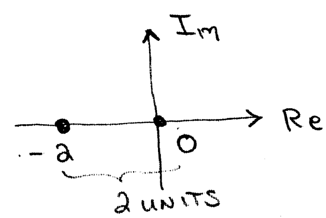
CENTER AT $X = 0$

(a) $(2x+4)y'' + 3xy' - 7(x+1)y = 0$

$y'' + \frac{3x}{2x+4}y' - \frac{7(x+1)}{2x+4}y = 0$

$2x + 4 = 0 \Rightarrow x = -2$

ONE SINGULAR PT: $X = -2$

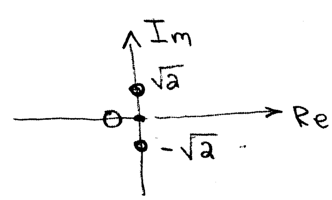


$\boxed{R \geq 2}$

(b) $(x^2+2)y'' + 9y' + x^2y = 0$

$y'' + \frac{9}{x^2+2}y' + \frac{x^2}{x^2+2}y = 0$

$x^2 + 2 = 0 \Rightarrow X = \pm\sqrt{2}\,i$



$\boxed{R \geq \sqrt{2}}$

TWO SINGULAR PTS: $X = \sqrt{2}\,i$

$X = -\sqrt{2}\,i$

7. (12 points) State the recurrence relation that describes the coefficients of the power series solution (centered at $x = 0$), and state the guaranteed radius of convergence (according to our theorem from class).

$$3y'' + xy' - 4y = 0$$

$$y'' + \frac{x}{3}y' - \frac{4}{3}y = 0$$

ANALYTIC EVERYWHERE. $\Rightarrow$ $\boxed{R = \infty}$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$$

REPLACE n WITH $n+2$

$$= \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$$

$$= \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[3(n+2)(n+1)a_{n+2} + (n-4)a_n\right] x^n$$

$$3(n+2)(n+1)a_{n+2} + (n-4)a_n = 0; \quad n = 0, 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{(4-n)a_n}{3(n+2)(n+1)}; \quad n = 0, 1, 2, \ldots$$

a_0 AND a_1 ARE ARBITRARY

Follow-up: Based on your recurrence relation, briefly explain how you would obtain the two desired linearly independent solutions.

NATURAL SPLIT EVEN/ODD.

FOR $y_1(x)$, USE THE EVEN VALUES OF n:

$$y_1(x) = a_0 + a_2 x^2 + a_4 x^4 + \ldots$$

FOR $y_2(x)$, USE THE ODD VALUES OF n:

$$y_2(x) = a_1 x + a_3 x^3 + a_5 x^5 + \ldots$$

The following problems due November 14, 2023. You must work on your own.

8. (5 points) Use the integral definition of the Laplace transform to find the transform of $f(t) = e^{-5t}$.

$$F(s) = \int_0^\infty e^{-st} e^{-5t}\, dt = \int_0^\infty e^{-(s+5)t}\, dt$$

$$= -\frac{1}{s+5} e^{-(s+5)t}\Big|_{t=0}^{t\to\infty} = \frac{1}{s+5} e^{-(s+5)t}\Big|_{t\to\infty}^{t=0}$$

$$= \frac{1}{s+5} - \frac{1}{s+5} \underbrace{\lim_{t\to\infty} e^{-(s+5)t}}_{\substack{\text{LIMIT} = 0 \\ \text{IF } s > -5}} = \frac{1}{s+5}, \quad s > -5$$

$$F(s) = \frac{1}{s+5}, \quad s > -5$$

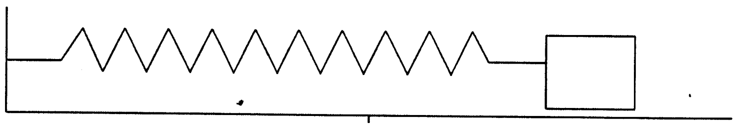
9. (5 points) Use the integral definition of the Laplace transform to find the transform of
$f(t) = \begin{cases} 2, & 0 \le t \le 10 \\ 0, & t > 10 \end{cases}$

$$F(s) = \int_0^{10} 2e^{-st}\, dt = -\frac{2}{s} e^{-st}\Big|_0^{10}$$

$$= \frac{2}{s} e^{-st}\Big|_{10}^{0} = \frac{2}{s} - \frac{2e^{-10s}}{s}$$

$$F(s) = \frac{2}{s} - \frac{2}{s e^{10s}}, \quad s \neq 0$$

10. (10 points) A 2-kg mass is attached to a spring with spring constant 6 N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 1 m to the **left** of equilibrium (compressing the spring) and pushed to the **left** at 2 m/sec. At the moment the mass is pushed, the periodic external force $F(t) = 4\cos 6t$ is applied.



$x = 0$
(Equilibrium)

(a) Set up the initial value problem that describes the motion of the mass.

$$2x'' + x' + 6x = 4\cos 6t, \quad x(0) = -1, \quad x'(0) = -2$$

(b) Use SageMath (or some other CAS) to solve the initial value problem. (If you need help with the SageMath syntax, see the posted lecture notes for section 2.6.)

SEE ATTACHED SHEET.

(c) Use SageMath (or some other CAS) to graph your solution for $0 \le t \le 25$. Attach a copy of the graph.

SEE ATTACHED SHEET.

(d) On your graph, indicate where the transient part of the solution is dominant and where the steady-state part is dominant.

SEE ATTACHED SHEET.

(e) Compute the gain factor for this system.

$$G = \frac{1}{\sqrt{(6-72)^2 + 36}} = \frac{1}{6\sqrt{122}} \approx 0.015$$

(f) Compute the resonance frequency for this system.

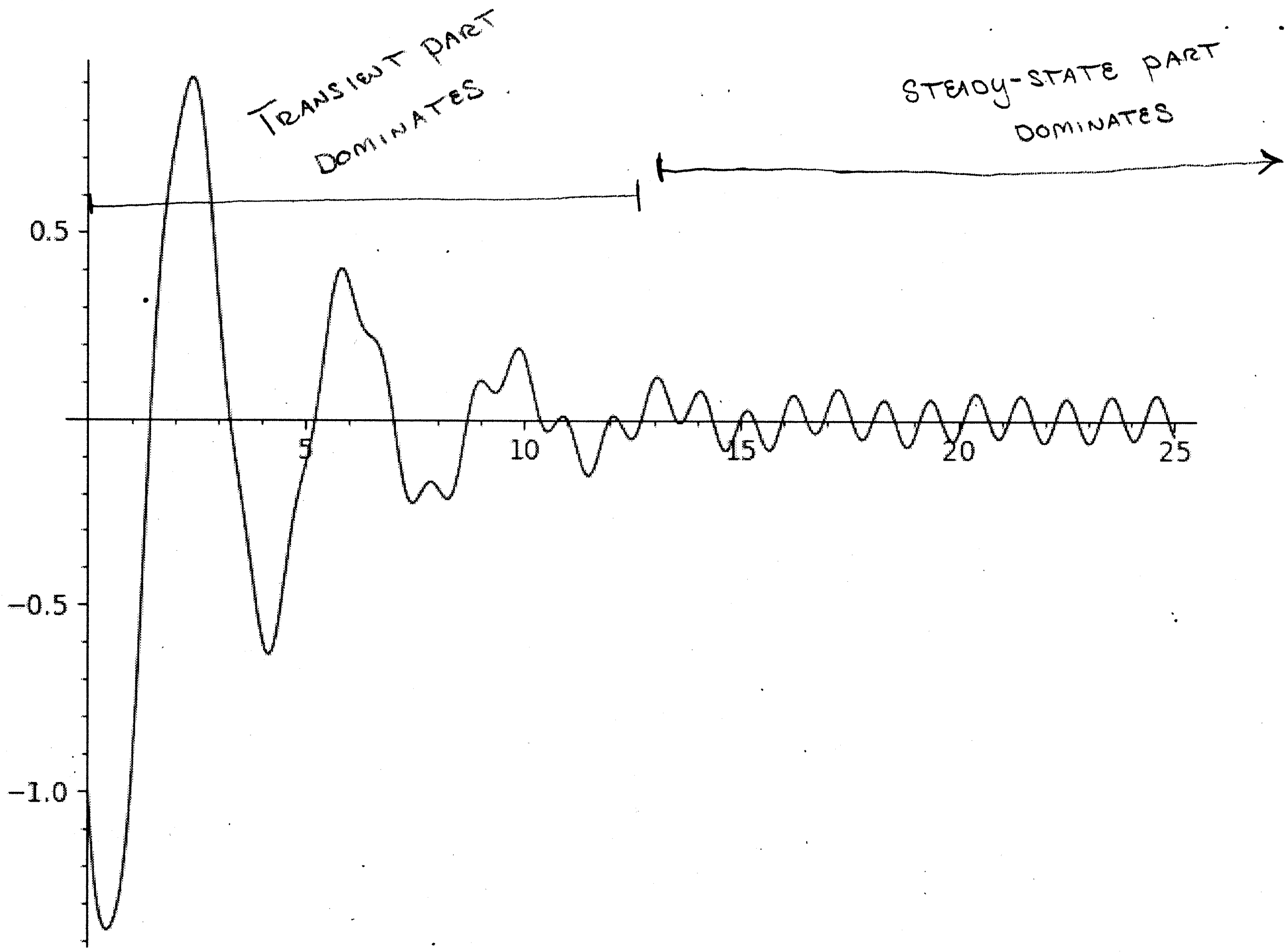
$$\gamma_r = \sqrt{\frac{6}{2} - \frac{1}{8}} = \sqrt{\frac{23}{8}} \approx 1.7$$

↑ NOT CLOSE TO $\gamma = 6$.
WE EXPECT SMALL GAIN.

```
t=var("t")
x=function("x")(t)
de=2*diff(x,t,2)+diff(x,t)+6*x==4*cos(6*t)
desolve(de,x,[0,-1,-2])
```

Solution is

```
x(t) = -4/8601*(415*sqrt(47)*sin(1/4*sqrt(47)*t) +
2021*cos(1/4*sqrt(47)*t))*e^(-1/4*t) - 11/183*cos(6*t) + 1/183*sin(6*t)
```



$$y'' - \frac{7x}{x^2+3} y' + \frac{16}{x^2+3} y = 0$$

$x^2 + 3 = 0 \Rightarrow x = \pm\sqrt{3}\,i$ DISTANCE FROM $\sqrt{3}\,i$ TO ZERO IS $\sqrt{3}$ UNITS

11. (16 points) State the recurrence relation that describes the coefficients of the power series solution (centered at $x = 0$), and state the guaranteed radius of convergence (according to our theorem from class).

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (x^2+3)y'' - 7xy' + 16y = 0$$

$$\boxed{R \geq \sqrt{3}}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n$$

(CAN START AT $n=0$) REPLACE n WITH $n+2$ (CAN START AT $n=0$)

$$= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[n(n-1)a_n + 3(n+2)(n+1)a_{n+2} - 7na_n + 16a_n \right] x^n$$

$$\Rightarrow (n^2 - 8n + 16)a_n + 3(n+2)(n+1)a_{n+2} = 0; \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \boxed{a_{n+2} = \frac{-(n-4)^2 a_n}{3(n+2)(n+1)}; \quad n = 0, 1, 2, 3, \ldots \quad a_0 \text{ \& } a_1 \text{ ARE ARBITRARY.}}$$

Follow-up: Write the first four terms of each of two linearly independent solutions.

$a_0 = 1, \; a_1 = 0$

$a_2 = -\frac{16}{6} = -\frac{8}{3}$

$a_4 = \frac{-4}{36} \cdot \left(-\frac{8}{3}\right) = \frac{8}{27}$

$a_6 = 0$

$$\boxed{y_1(x) = 1 - \frac{8}{3}x^2 + \frac{8}{27}x^4}$$

$a_0 = 0, \; a_1 = 1$

$a_3 = \frac{-9}{18} = -\frac{1}{2}$

$a_5 = \frac{-1}{60} \cdot \left(-\frac{1}{2}\right) = \frac{1}{120}$

$a_7 = \frac{-1}{126} \cdot \left(\frac{1}{120}\right) = \frac{-1}{15120}$

$$\boxed{y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{120}x^5 - \frac{1}{15120}x^7 + \cdots}$$