Math 240 - Final Exam A

December 8, 2023

Name Key

Score

Show all work to receive full credit. You must work individually. This test is due December 14. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD. All integration must be done by hand.

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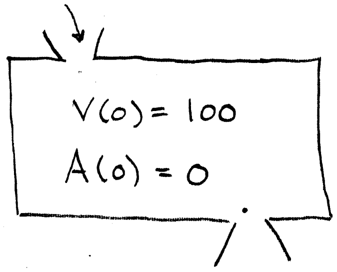
1. (10 points) A brine solution with a salt concentration of 0.2 kg/L flows into a large tank at the constant rate of 4 L/min. The tank initially held 100 L of pure water. The solution inside the tank is well-mixed and flows out at the rate of 2 L/min. Determine the mass of salt in the tank after t minutes. When will the mass of salt in the tank reach 40 kg?

$0.2 \frac{kg}{L} \times 4 \frac{L}{min} = 0.8 \frac{kg}{min}$



$A(t)$ = MASS OF SALT IN TANK AT TIME t

$V(t)$ = VOLUME AT TIME t = $100 + 2t$

$$\frac{A(t)}{V(t)} \frac{kg}{L} \times 2 \frac{L}{min} = \frac{2A}{100+2t} \frac{kg}{min}$$

$$\frac{dA}{dt} = 0.8 - \frac{A}{50+t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{50+t} A = 0.8$$

$$\mu(t) = e^{\int \frac{1}{50+t} dt} = e^{\ln|50+t|} = (50+t)$$

$$\mu(t) \cdot A(t) = \int \mu(t)(0.8)\, dt$$

$$(50+t)\, A(t) = \int (0.8)(50+t)\, dt$$

$$= 40t + 0.4t^2 + C$$

$A(0) = 0 \Rightarrow C = 0$

$$\boxed{A(t) = \frac{40t + 0.4t^2}{50+t}, \quad t \geq 0}$$

$$40 = \frac{40t + 0.4t^2}{50+t}$$

$$\Rightarrow 0.4t^2 = 2000$$

$$t^2 = 5000$$

$$\boxed{t = 50\sqrt{2} \text{ min} \approx 70.7 \text{ min}}$$

2. (10 points) Use Laplace transform methods to solve the following equation.

$$y'' + 4y' - 5y = te^t, \quad y(0) = 1,\ y'(0) = 0$$

$$s^2 Y - s + 4\left[sY - 1\right] - 5Y = \frac{1}{(s-1)^2}$$

$$(s^2+4s-5)Y = \frac{1}{(s-1)^2} + s + 4$$

$$Y(s) = \underbrace{\frac{\frac{1}{(s-1)^2} + s + 4}{s^2+4s-5}}_{\substack{\text{Computer Alg System} \\ \text{PFD}}} = \frac{35}{216}\frac{1}{s+5} + \frac{181}{216}\frac{1}{x-1} - \frac{1}{36}\frac{1}{(x-1)^2} + \underbrace{\frac{1}{6}\frac{1}{(x-1)^3}}_{\frac{1}{12}\frac{2}{(x-1)^3}}$$

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t}$$

3. (10 points) Solve the following initial value problem. (It may be helpful to use a property of logarithms.)

WE'RE AUTOMATICALLY ASSUMING $x, y > 0$

$$\frac{dy}{dx} = \frac{y(1+\ln y - \ln x)}{x}, \quad y(1) = 8$$

$$\frac{dy}{dx} = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right) = \frac{y}{x} + \frac{y}{x}\ln\left(\frac{y}{x}\right)$$

$$u = \frac{y}{x} \Rightarrow ux \Rightarrow y = \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$x\frac{du}{dx} = u\ln u \Rightarrow \frac{du}{u\ln u} = \frac{1}{x}dx \Rightarrow \int \frac{1}{u\ln u}du = \int \frac{1}{x}dx$$

$w = \ln u$

$dw = \frac{1}{u}du$

$$\int \frac{1}{w}dw = \int \frac{1}{x}dx$$

$$\ln|w| = \ln|x| + C_1$$

$$w = C_2 x$$

$$\ln u = C_2 x \Rightarrow u = e^{C_2 x}$$

$$\frac{y}{x} = e^{C_2 x} \Rightarrow y(x) = xe^{C_2 x}$$

$$y(1) = 8 \Rightarrow 8 = e^{C_2} \Rightarrow C_2 = \ln 8$$

$$y(x) = x\left(e^{\ln 8}\right)^x = x8^x$$

$$\boxed{y(x) = x8^x, \quad x > 0}$$

4. (10 points) Consider the linear, 2nd-order equation $xy'' + y' + xy = 0$. This equation is called *Bessel's equation of order 0*. Explain why it is probably not a good idea to look for a power series solution centered at $x = 0$. Nonetheless, find such a power series solution. Contrary to our typical problems of this form, you should have only one arbitrary constant of integration. Write out several terms of the solution that satisfies $y(0) = 1$. (The solution is typically called $J_0(x)$, and it is defined in SageMath by `bessel_J(0,x)`. You'll get one point extra credit if you attach a graph of $J_0(x)$ for $0 \le x \le 25$.)

SINCE $X = 0$ IS A SINGULAR POINT OF THE EQUATION, WE CANNOT IMMEDIATELY BE CERTAIN THAT A POWER SERIES SOLUTION AT $X=0$ WILL CONVERGE. LET'S GO FOR IT ANYWAY.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = xy'' + y' + xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$n \to n+1 \qquad n \to n+1 \qquad n \to n-1$$

$$= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= a_1 + \sum_{n=1}^{\infty} \left[(n+1) n a_{n+1} + (n+1) a_{n+1} + a_{n-1} \right] x^n$$

$$\Rightarrow a_1 = 0 \quad \text{AND} \quad (n^2 + 2n + 1) a_{n+1} + a_{n-1} = 0; \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow a_1 = 0 \quad \text{AND} \quad a_{n+1} = \frac{-1}{(n+1)^2} a_{n-1}; \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow a_0 = \text{ARBITRARY}, \quad a_1 = 0, \quad a_2 = \frac{-a_0}{2^2}, \quad a_3 = 0$$

$$a_4 = \frac{a_0}{2^2 4^2}, \quad a_5 = 0, \quad a_6 = \frac{-a_0}{2^2 4^2 6^2}, \quad a_7 = 0 \ldots$$

$$y(x) = a_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \frac{x^8}{2^2 4^2 6^2 8^2} - \cdots \right.$$

$$y(0) = 1 \Rightarrow a_0 = 1$$

THIS IS ONLY ONE OF TWO NECESSARY LIN. INDEP. SOLUTIONS.

5. (10 points) Solve the following one-dimensional heat equation with Neumann boundary conditions. Do not derive the solution method–just use the theorem in the textbook.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

$L = 1$

$$u_x(0,t) = u_x(1,t) = 0,$$

$k = 2$

$$u(x,0) = x - x^2$$

$$f(x) = x - x^2$$

THE SOLUTION IS $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-2\pi^2 n^2 t} \cdot \cos(n\pi x)$

WHERE $a_n = n^{TH}$ FOURIER COSINE SERIES COEFFICIENT FOR $f(x) = x - x^2$

$$a_0 = 2\int_0^1 (x - x^2)\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

$$a_n = 2\int_0^1 (x - x^2)\cos(n\pi x)\,dx = 2\left[\left(0 - \frac{1}{n^2\pi^2}\cos(n\pi) + 0\right) - \left(0 + \frac{1}{n^2\pi^2} + 0\right)\right]$$

+	$(x - x^2)$	$\cos(n\pi x)$
−	$(1 - 2x)$	$\frac{1}{n\pi}\sin(n\pi x)$
+	(-2)	$-\frac{1}{n^2\pi^2}\cos(n\pi x)$
−	0	$-\frac{1}{n^3\pi^3}\sin(n\pi x)$

$$= \frac{-2}{n^2\pi^2}\left(\cos(n\pi) + 1\right)$$

$$= \frac{-2}{n^2\pi^2}\left((-1)^n + 1\right) \; ; \quad n = 1, 2, 3, 4, \ldots$$

$$u(x,t) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{-2\left[(-1)^n + 1\right]}{n^2\pi^2} e^{-2\pi^2 n^2 t} \cos(n\pi x)$$