Math 240 - Final Exam A

Name ____________________________

December 8, 2023

Score ___________

Show all work to receive full credit. You must work individually. This test is due December 14. If your approach to any problem on the test requires a partial fraction decomposition, you may use technology to find your PFD. All integration must be done by hand.

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1. (10 points) A brine solution with a salt concentration of $0.2\,\mathrm{kg/L}$ flows into a large tank at the constant rate of $4\,\mathrm{L/min}$. The tank initially held $100\,\mathrm{L}$ of pure water. The solution inside the tank is well-mixed and flows out at the rate of $2\,\mathrm{L/min}$. Determine the mass of salt in the tank after t minutes. When will the mass of salt in the tank reach $40\,\mathrm{kg}$?

2. (10 points) Use Laplace transform methods to solve the following equation.

$$y'' + 4y' - 5y = te^t, \quad y(0) = 1,\, y'(0) = 0$$

3. (10 points) Solve the following initial value problem. (It may be helpful to use a property of logarithms.)

$$\frac{dy}{dx} = \frac{y(1 + \ln y - \ln x)}{x}, \quad y(1) = 8$$

4. (10 points) Consider the linear, 2nd-order equation $xy'' + y' + xy = 0$. This equation is called *Bessel's equation of order 0*. Explain why it is probably not a good idea to look for a power series solution centered at $x = 0$. Nonetheless, find such a power series solution. Contrary to our typical problems of this form, you should have only one arbitrary constant of integration. Write out several terms of the solution that satisfies $y(0) = 1$. (The solution is typically called $J_0(x)$, and it is defined in SageMath by `bessel_J(0,x)`. You'll get one point extra credit if you attach a graph of $J_0(x)$ for $0 \leq x \leq 25$.)

5. (10 points) Solve the following one-dimensional heat equation with Neumann boundary conditions. Do not derive the solution method–just use the theorem in the textbook.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

$$u_x(0,t) = u_x(1,t) = 0,$$

$$u(x,0) = x - x^2$$