Math 240 - Final Exam B

December 14, 2023

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

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1. (4 points) Use any method to find the general solution of $y''' + 2y'' + 4y' = 0$.

$r^3 + 2r^2 + 4r = 0$

$r(r^2 + 2r + 4) = 0$

$r = 0 \qquad r^2 + 2r + 1 = -3$

$(r+1)^2 = -3$

$r = -1 \pm i\sqrt{3}$

$$\boxed{y(x) = c_1 + c_2 e^{-x}\cos(\sqrt{3}x) + c_3 e^{-x}\sin(\sqrt{3}x)}$$

2. (8 points) Use any method to find the general solution of $y'' - 2y' + y = 8e^t$.

Homo. Eqn:

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1,\ r = 1$

$y_c(t) = c_1 e^t + c_2 t e^t$

NonHomo. Eqn:

$g(t) = 8e^t$

$y_p(t) = t^s(Ae^t)$

$s = 2$

$y_p(t) = At^2e^t$

$y_p'(t) = At^2e^t + 2Ate^t$

$y_p''(t) = At^2e^t + 4Ate^t + 2Ae^t$

$y_p'' - 2y_p' + y_p = 8e^t$

$= 2Ae^t \Rightarrow A = 4$

$$\boxed{y(t) = c_1e^t + c_2te^t + 4t^2e^t}$$

3. (10 points) In this problem, you will find five (5) ordinary differential equations. Each equation has a specific name or can be described by a word, phrase, or short sentence. For each equation, write that name or description, and then write a sentence describing a solution method. Be brief, but specific, when describing your solution method.

(a) $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x, \quad x > 0$

$$\frac{dy}{dx} - \frac{2}{x}y = \underbrace{x^2 \cos x}_{q(x)}$$

1ST ORDER LINEAR. SOLUTION METHOD: COMPUTE INTEGRATING FACTOR, $\mu(x)$. THEN SOLUTION FOLLOWS FROM

$$\mu(x)\,y(x) = \int \mu(x)\,q(x)\,dx$$

(b) $3xy^2\frac{dy}{dx} + y^3 = x^3$

$$\frac{dy}{dx} + \frac{1}{3x}y = \frac{x^2}{3y^2}$$

THIS EQUATION IS BERNOULLI. (IT IS ALSO HOMOGENEOUS AND EXACT.)

TO SOLVE AS BERNOULLI, REWRITE $3y^2\frac{dy}{dx} + \frac{1}{x}y^3 = x^2$.

THEN $u = y^3$ WILL TRANSFORM IT TO LINEAR.

(c) $y'' - 4y' + 4y = t^3e^{2t}; \quad y(0) = 0,\ y'(0) = 0$

2ND ORDER, NONHOMOGENEOUS, CONST. COEFF. LINEAR, INITIAL VALUE PROBLEM

USE LAPLACE TRANSFORMS. OR SOLVE THE CORRESPONDING HOMO. EQN. BY USING CHAR. EQUATION, AND THEN USE UNDET. COEFFS.

(d) $\underbrace{(x + xy^2)}_{M}\,dx + \underbrace{x^2y}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 2xy = \frac{\partial N}{\partial x}$$

THIS EQUATION IS EXACT.

SOLVE BY USING $\int M\,dx = \int N\,dy$

(IT IS ALSO SEPARABLE: $\frac{dy}{dx} = \frac{1}{x}\left(\frac{1+y^2}{y}\right)$.)

(AND ALSO BERNOULLI: $\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x}y^{-1}$)

(e) $x'' - tx' + x = 0; \quad x(0) = 0,\ x'(0) = 0$

2ND ORDER, LINEAR, HOMOGENEOUS, BUT NOT CONST. COEFF.

NO VERY SPECIAL FORM.

SOLVE BY POWER SERIES, $x = \sum_{n=0}^{\infty} a_n t^n$.

4. (16 points) Choose any two of the equations from problem 3 and solve each by using the solution method that you described above.

(a) First problem:

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 \cos x, \quad x > 0$$

$$\mu(x) = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = \frac{1}{x^2}$$

$$\frac{1}{x^2} y(x) = \int \frac{1}{x^2} (x^2 \cos x)\, dx$$

$$\frac{1}{x^2} y(x) = \int \cos x \, dx = \sin x + C$$

$$\boxed{y(x) = x^2 \sin x + Cx^2}$$

(b) Second problem:

$$3xy^2 \frac{dy}{dx} + y^3 = x^3$$

$$3y^2 \frac{dy}{dx} + \frac{1}{x} y^3 = x^2 , \quad x \neq 0$$

$$u = y^3 \quad \frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{1}{x} u = x^2$$

Assuming

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln |x|} = x , \quad x > 0$$

$$x\, u(x) = \int x^3 \, dx$$

$$x\, u(x) = \frac{1}{4} x^4 + C$$

$$u(x) = \frac{1}{4} x^3 + \frac{C}{x}$$

(c) $y'' - 4y' + 4y = t^3 e^{2t}$, $y(0) = y'(0) = 0$

$$s^2 Y - 4sY + 4Y = \frac{6}{(s-2)^4}$$

$$(s^2 - 4s + 4)Y = \frac{6}{(s-2)^4}$$

$$(s-2)^2 Y = \frac{6}{(s-2)^4}$$

$$Y(s) = \frac{6}{(s-2)^6} = \frac{6}{5!} \frac{5!}{(s-2)^6} = \frac{1}{20} \frac{5!}{(s-2)^6}$$

$$\boxed{y(t) = \frac{1}{20} t^5 e^{2t}}$$

(d) $(x+xy^2)\,dx + x^2y\,dy = 0$

$$F_x(x,y) = x + xy^2 \Rightarrow F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}x^2y^2 + g(y)$$

$$F_y(x,y) = x^2y \Rightarrow F(x,y) = \frac{1}{2}x^2y^2 + h(x)$$

$$F(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}x^2$$

SOLUTION IS

$$\frac{1}{2}x^2y^2 + \frac{1}{2}x^2 = C_1$$

- OR -

$$y^2 = \frac{2C_1 - x^2}{x^2}$$

- OR -

$$y(x) = \sqrt{\frac{C - x^2}{x^2}}$$

(e) $x'' - tx' + x = 0; \quad x(0) = x'(0) = 0$

$$x = \sum_{n=0}^{\infty} a_n t^n, \quad x' = \sum_{n=1}^{\infty} n a_n t^{n-1}, \quad x'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$0 = x'' - tx' + x$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - \sum_{n=1}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n$$

$n \to n+2$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (1-n) a_n \right] t^n$$

$$\Rightarrow \quad (n+2)(n+1) a_{n+2} + (1-n) a_n = 0; \quad n = 0, 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{n-1}{(n+2)(n+1)} a_n; \quad n = 0, 1, 2, 3, \ldots$$

a_0 & a_1 ARE ARBITRARY

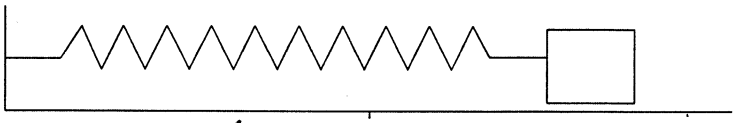
$x(0) = 0 \Rightarrow a_0 = 0$

$x'(0) = 0 \Rightarrow a_1 = 0$

So ALL a_n's ARE ZERO!

$$\boxed{x(t) = 0}$$

5. (12 points) A 1-kg mass is attached to a spring with spring constant $k = 25\,\mathrm{N/m}$. The damping constant for the system is $b = 6\,\mathrm{N\text{-}sec/m}$. The mass is moved 1 m to the right of equilibrium (stretching the spring) and released from rest. Find the equation of motion. If applicable, write your solution in terms of a single sine or cosine with a phase shift.



$x = 0$
(Equilibrium)

$$x'' + 6x' + 25x = 0; \quad x(0) = 1, \; x'(0) = 0$$

$$r^2 + 6r + 25 = 0$$

$$(r+3)^2 = -16$$

$$r = -3 \pm 4i.$$

$$x(t) = c_1 e^{-3t} \cos 4t + c_2 e^{-3t} \sin 4t$$

$$x(0) = 1 \Rightarrow c_1 = 1$$

$$x'(0) = 0 \Rightarrow -3c_1 + 4c_2 = 0$$

$$c_2 = \frac{3}{4}$$

$$x(t) = e^{-3t} \cos 4t + \frac{3}{4} e^{-3t} \sin 4t$$

$$A = \sqrt{1^2 + \left(\tfrac{3}{4}\right)^2} = \frac{5}{4}$$

$$c_1 = 1 = \frac{5}{4} \sin \varphi$$

$$c_2 = \frac{3}{4} = \frac{5}{4} \cos \varphi$$

φ IN QUAD I AND $\tan \varphi = \frac{4}{3}$

$$x(t) = \frac{5}{4} e^{-3t} \sin\left(4t + \tan^{-1}\left(\tfrac{4}{3}\right)\right)$$

$\square$

Follow-up: After the mass passes through equilibrium for the first time, it will very soon reach its farthest point to the left of equilibrium. When will it reach that point?

$$x'(t) = 0 \Rightarrow -\frac{15}{4} e^{-3t} \sin(\square) + 5e^{-3t} \cos(\square) = 0$$

$$\Rightarrow 5 \cos(\square) = \frac{15}{4} \sin(\square)$$

$$\Rightarrow \frac{20}{15} = \frac{4}{3} = \tan(\square)$$

$$\Rightarrow 4t + \cancel{\tan^{-1}\left(\tfrac{4}{3}\right)} = \cancel{\tan^{-1}\left(\tfrac{4}{3}\right)} + k\pi$$

WITH $k = 1$, WE HAVE

$$t = \frac{\pi}{4} \text{ SEC}$$