MTH 240 Assignment #1 Key

1.) $$y(x) = \frac{c \sin x}{x^2}$$

$$y'(x) = \frac{cx^2 \cos x - 2cx \sin x}{x^4} = \frac{cx \cos x - 2c \sin x}{x^3}$$

$$y''(x) = \frac{x^3\left(c \cos x - cx \sin x - 2c \cos x\right) - 3x^2\left(cx \cos x - 2c \sin x\right)}{x^6}$$

$$= \frac{cx\left(-\cos x - x \sin x\right) - 3c\left(x \cos x - 2 \sin x\right)}{x^4}$$

$$= \frac{-4cx \cos x - cx^2 \sin x + 6c \sin x}{x^4}$$

$$x^2y'' + 4xy' + (x^2+2)y =$$

$$\frac{-\cancel{4cx \cos x} - \cancel{cx^2 \sin x} + \cancel{6c \sin x}}{x^2} + \frac{\cancel{4cx \cos x} - \cancel{8c \sin x}}{x^2} + \frac{\cancel{cx^2 \sin x} + \cancel{2c \sin x}}{x^2}$$

$$= \frac{0}{x^2} = 0 \checkmark$$

THIS IS TRUE REGARDLESS OF THE VALUE OF THE CONSTANT C.

2.) $y(x) = ay_1(x) + by_2(x)$

$y'(x) = ay_1'(x) + by_2'(x)$

$y''(x) = ay_1''(x) + by_2''(x)$

$$x^2y'' + 4xy' + (x^2+2)y =$$

$$x^2(ay_1'' + by_2'') + 4x(ay_1' + by_2') + (x^2+2)(ay_1 + by_2) =$$

$$a\Big[\underbrace{x^2y_1'' + 4xy_1' + (x^2+2)y_1}_{\text{THIS IS ZERO BECAUSE } y_1 \text{ IS A SOLUTION.}}\Big] + b\Big[\underbrace{x^2y_2'' + 4xy_2' + (x^2+2)y_2}_{\text{THIS IS ZERO BECAUSE } y_2 \text{ IS A SOLUTION.}}\Big] =$$

$$a[0] + b[0] = 0 \quad \checkmark$$

THIS IS TRUE FOR ANY CONSTANTS a AND b.

3.) THE ONLY SOLUTION OF $|y'| + |y| = 0$ IS THE CONSTANT FUNCTION $y(x) \equiv 0$.

ANY NONZERO FUNCTION WOULD HAVE POSITIVE ABSOLUTE VALUES, AND WE COULD NOT GET A ZERO SUM.

4.) FIRST, IT'S INTUITIVELY CLEAR THAT k SHOULD BE NEGATIVE BECAUSE ATMOSPHERIC PRESSURE DECREASES WITH ALTITUDE. AS ALTITUDE INCREASES AND PRESSURE DECREASES, THE THINNING ATMOSPHERE WILL CAUSE THE PRESSURE TO DECREASE FURTHER. SO A MODEL THAT SUGGESTS THE RATE OF CHANGE OF PRESSURE IS PROPORTIONAL TO PRESSURE SEEMS LIKE A GOOD PLACE TO START. SUCH A MODEL WOULD SUGGEST THAT PRESSURE DECREASES EXPONENTIALLY WITH ALTITUDE. DATA SUPPORTS THIS IDEA.

$$p(a) = p_0 e^{ka} \Rightarrow \frac{dp}{da} = p_0 k e^{ka} = k p_0 e^{ka} = kp \quad \checkmark$$

WE ALSO HAVE $p(0) = p_0 e^{k(0)} = p_0 e^0 = p_0$. $\checkmark$

5.) $$\frac{d^2u}{dw^2} + 5\left(\frac{du}{dw}\right)^3 - 4u = w$$

DEPENDENT VARIABLE IS u.

INDEP. VARIABLE IS w.

EQUATION IS ORDINARY. NONLINEAR BECAUSE OF $\left(\frac{du}{dw}\right)^3$.

THE EQUATION IS 2ND ORDER.

6.) THE EQUATION IS <u>NONLINEAR</u> BECAUSE A FUNCTION OF y (e^y) IS MULTIPLIED BY $\frac{dy}{dx}$: $e^y y'$.

$$y = \ln(x+C) \Rightarrow y' = \frac{1}{x+C}$$

$$e^y y' = e^{\ln(x+C)}\left(\frac{1}{x+C}\right) = \cancel{(x+C)}\left(\frac{1}{\cancel{x+C}}\right) = 1 \checkmark$$

$$y(0) = 0 \Rightarrow \ln(C) = 0$$

$$\Rightarrow \boxed{C = 1}$$

$$\boxed{y(x) = \ln(x+1)}$$

7.) $\frac{dy}{dx} = x^2 \sin \pi x, \quad y\left(\frac{1}{2}\right) = 2$

$$y = \int x^2 \sin \pi x \, dx = -\frac{x^2}{\pi}\cos \pi x + \frac{2x}{\pi^2}\sin \pi x + \frac{2}{\pi^3}\cos \pi x + C$$

SIGNS	u AND DERIVS	$\frac{dv}{dx}$ & ANTIS
$+$	x^2	$\sin \pi x$
$-$	$2x$	$-\frac{1}{\pi}\cos \pi x$
$+$	2	$-\frac{1}{\pi^2}\sin \pi x$
$-$	0	$\frac{1}{\pi^3}\cos \pi x$

$$y\left(\frac{1}{2}\right) = 2$$

$$\Rightarrow 0 + \frac{1}{\pi^2} + 0 + C = 2$$

$$\Rightarrow C = 2 - \frac{1}{\pi^2}$$

$$\boxed{y(x) = -\frac{x^2}{\pi}\cos \pi x + \frac{2x}{\pi^2}\sin \pi x + \frac{2}{\pi^3}\cos \pi x + 2 - \frac{1}{\pi^2}}$$

8.) $\frac{dN}{dt} = kN(P-N)$

9.) $y = g(x)$

$$\text{SLOPE OF NORMAL LINE} = \frac{-1}{dy/dx} = \frac{y-1}{x-\frac{x}{3}} = \frac{y-1}{\frac{2x}{3}} = \frac{3y-3}{2x}$$

$$-\frac{dy}{dx} = \frac{2x}{3y-3}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{3-3y}}$$

10.) $\frac{dy}{dx} = \frac{x^3}{\sqrt{x^4+15}}$, $y(1) = 5$

$$y = \int \frac{x^3}{\sqrt{x^4+15}}\,dx = \frac{1}{4}\int u^{-1/2}\,du = \frac{1}{2}u^{1/2} + C$$

$$= \frac{1}{2}\sqrt{x^4+15} + C$$

$u = x^4 + 15$

$du = 4x^3\,dx$

$\frac{1}{4}\,du = x^3\,dx$

$y(1) = 5$

$\Rightarrow \frac{1}{2}\sqrt{16} + C$

$= 2 + C = 5$

$\Rightarrow C = 3$

$$\boxed{y(x) = \frac{1}{2}\sqrt{x^4+15} + 3}$$