MTH 240 Assignment #2 Key

1) $\frac{dP}{dt} = P(P-1)(2-P)$, P IN THOUSANDS. SEE ATTACHED SLOPE FIELD. (NEXT PAGE)

a) $P(0) = 4$

EVEN WITHOUT LOOKING AT THE SLOPE FIELD...

$P = 4 \Rightarrow dP/dt = -24$. THIS IS AN INITIAL DECREASE OF 24000 PER TIME.

THIS IS A VERY FAST DECREASE, AND THIS AGREES WITH EVIDENCE FROM THE SLOPE FIELD.

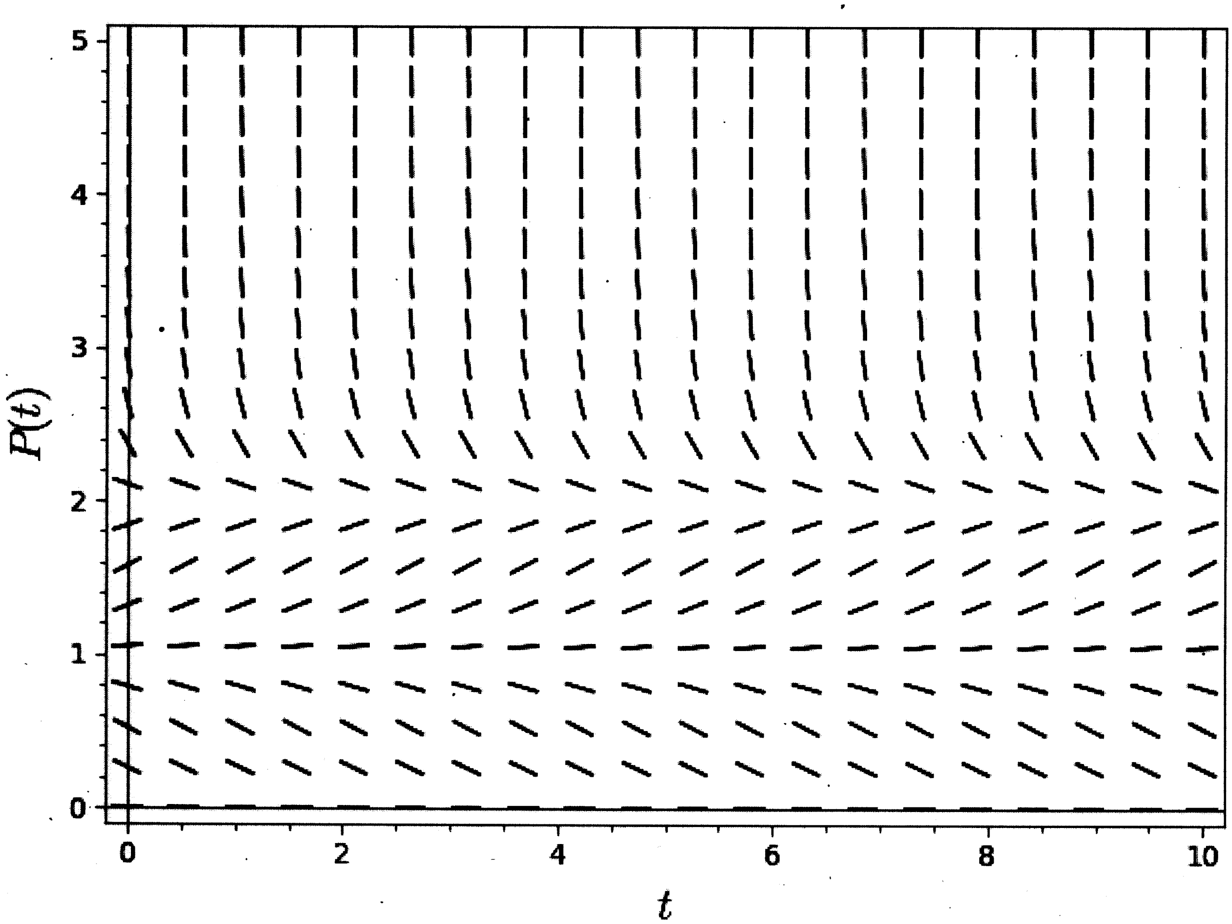
b) $P(0) = 4 \Rightarrow \boxed{P(t) \to 2}$ (THAT IS, 2 THOUSAND). AS $t \to \infty$

c) $P(0) = 1.5 \Rightarrow \boxed{P(t) \to 2}$ AS $t \to \infty$

d) $P(0) = 2 \Rightarrow \boxed{P(t) \equiv 2}$ (CONSTANT)

e) $P(0) < 1 \Rightarrow \boxed{P(t) \to 0}$ AS $t \to \infty$

Slope Field for Problem 1.



2) $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}\,,\quad y(8) = 1$

$f(x,y) = 3x - \sqrt[3]{y-1}$ f IS CONTINUOUS FOR ALL (x,y)

$f_y(x,y) = -\frac{1}{3}(y-1)^{-2/3}$ f_y IS CONTINUOUS EVERYWHERE

$= \dfrac{-1}{3\sqrt[3]{(y-1)^2}}$ EXCEPT WHERE $y = 1$

(A) A SOLUTION EXISTS, BUT UNIQUENESS IS NOT GUARANTEED.

3) $\dfrac{dy}{dx} = \sqrt{xy}\,,\quad y(1) = 0$

$f(x,y) = \sqrt{xy}$ IS NOT CONTINUOUS ON ANY RECTANGLE CONTAINING $(1,0)$ BECAUSE SUCH A RECTANGLE WOULD HAVE POINTS WHERE $xy < 0$.

(C) A SOLUTION IS NOT GUARANTEED.

4) $(y')^2 - xy' + y = 0, \quad y(2) = 1$

$$y' = \frac{x \pm \sqrt{(-x)^2 - 4(1)(y)}}{2} = \frac{x \pm \sqrt{x^2 - 4y}}{2}$$

$$f(x,y) = \frac{x \pm \sqrt{x^2 - 4y}}{2}$$

$f(x,y)$ IS NOT CONTINUOUS AS A RECTANGLE AROUND $(2,1)$. BECAUSE SUCH A RECTANGLE WOULD CONTAIN POINTS WHERE

$x^2 - 4y < 0$.

(C) A SOLUTION IS NOT GUARANTEED.

5) $\frac{dy}{dx} = x\sqrt{y}, \quad y(1) = 4$

$f(x,y) = x\sqrt{y}$

$y_{n+1} = y_n + 0.1 x_n \sqrt{y_n}$

$y_0 = 4$

$x_0 = 1$

$y_1 = 4 + 0.1(1)(\sqrt{4})$

$= 4.2$

$x_1 = 1.1$

$y_2 = 4.2 + 0.1(1.1)(\sqrt{4.2})$

≈ 4.255433

$x_2 = 1.2$

$y_3 = 4.255433 + 0.1(1.2)(\sqrt{4.255433})$

≈ 4.677873

$x_3 = 1.3$

$y(1.3) \approx 4.677873$

Using TECHNOLOGY WITH $h = 0.01$,

$y(1.3) \approx 4.715471$.

6) $\frac{dy}{dx} = y^2 - 4$, $y(0) = -2$

$\frac{1}{y^2-4}\,dy = 1\,dx$ Assuming $y \neq 2, -2$.

But y is equal to -2.

$\boxed{y(x) \equiv -2 \text{ is a singular solution.}}$

Continuing by assuming $y \neq 2, -2 \ldots$

$$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

Using cover up,
$A = \frac{1}{4}$, $B = -\frac{1}{4}$

$$\left(\frac{\frac{1}{4}}{y-2} - \frac{\frac{1}{4}}{y+2}\right) dy = 1\,dx$$

$$\left(\frac{1}{y-2} - \frac{1}{y+2}\right) dy = 4\,dx$$

$$\ln|y-2| - \ln|y+2| = 4x + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C_1$$

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x}$$

$$\frac{y-2}{y+2} = C_3 e^{4x}$$

$$y - 2 = C_3 e^{4x}(y+2)$$

$$y - C_3 e^{4x} y = 2C_3 e^{4x} + 2$$

$$y(x) = \frac{2C_3 e^{4x} + 2}{1 - C_3 e^{4x}}$$

$$y(0) = \frac{2C_3 + 2}{1 - C_3} = -2$$

$$\Rightarrow \underbrace{2C_3 + 2 = -2 + 2C_3}_{\text{No such } C_3\,!}$$

But we already knew this

$y(x)$ is no good

for $y = 2$ or -2.

7) $$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{1}{T - T_s}\, dT = k\, dt$$

$$\ln|T - T_s| = kt + C_1$$

$$|T - T_s| = C_2 e^{kt}$$

$$T - T_s = C_3 e^{kt}$$

$$T(t) = T_s + C_3 e^{kt}$$

$$T_s = 70^\circ$$

$$T(0) = 300^\circ$$

$$T(3) = 200^\circ$$

$$T(t) = 70 + C_3 e^{kt}$$

$$T(0) = 300 = 70 + C_3$$

$$C_3 = 230$$

$$T(t) = 70 + 230e^{kt}$$

$$T(3) = 200 = 70 + 230e^{3k}$$

$$\frac{130}{230} = e^{3k}$$

$$k = \frac{\ln(13/23)}{3}$$

$$\boxed{T(t) = 70 + 230e^{\left(\frac{\ln 13/23}{3}\right)t}}$$

LET'S SAY "ALMOST" ROOM TEMP IS 70.1°

$$230e^{\left(\frac{\ln 13/23}{3}\right)t} = 0.1$$

$$\boxed{t \approx 40.7 \text{ min}}$$

8) $x^2 y' = y - xy$

$\frac{dy}{dx} = \frac{y(1-x)}{x^2}$, $x \neq 0$ (BASED ON INITIAL CONDITION, WE CAN ASSUME $x < 0$)

$\frac{1}{y}\, dy = \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$

$\ln|y| = -\frac{1}{x} - \ln|x| + C_1$

$|y| = C_2 e^{-\frac{1}{x} - \ln|x|} = C_2 e^{-\frac{1}{x}} e^{-\ln|x|} = \frac{C_2 e^{-1/x}}{|x|}$

$y(x) = \frac{C_3 e^{-1/x}}{|x|}$ $\quad y(-1) = -1 \Rightarrow -1 = C_3 e$

$C_3 = -\frac{1}{e}$

$$y(x) = \frac{-e^{-1/x}}{e|x|}$$

9) $\sin x \, dx = \frac{1 + 2y^2}{y} dy = \left(\frac{1}{y} + 2y\right) dy$

$\cos x = \ln|y| + y^2 + C$

$$x(y) = \cos^{-1}\left(\ln|y| + y^2 + C\right)$$