MTH 240 Assignment #3 Key

1) Solve. $\frac{dy}{dx} = 3y$ by using 1^{st}-order linear approach.

$$\frac{dy}{dx} - 3y = 0$$

$$\mu(x) = e^{\int -3dx} = e^{-3x}$$

$$e^{-3x} y = \int 0 \cdot e^{-3x} dx = \int 0\, dx = C$$

$$\boxed{y(x) = Ce^{3x}}$$

2)

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x, \quad x \neq 0$$

$$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = \frac{1}{x^4}$$

$$\frac{1}{x^4} y(x) = \int x e^x dx$$

$$u = x \qquad du = dx$$
$$dv = e^x dx \qquad v = e^x$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$\boxed{y(x) = x^5 e^x - x^4 e^x + Cx^4}$$

3) $y' + 2xy = x \,,\; y(0) = -3$

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2} y(x) = \int x e^{x^2}\,dx = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^{x^2} + C$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$y(x) = \frac{1}{2} + \frac{C}{e^{x^2}}$$

$$y(x) = \frac{1}{2} + Ce^{-x^2} \qquad y(0) = -3 \Rightarrow \frac{1}{2} + C = -3 \Rightarrow C = -\frac{7}{2}$$

$$y(x) = \frac{1}{2} - \frac{7}{2}e^{-x^2}$$

4) $\frac{dx}{dy} = x + y^2 \,,\; x(0) = -2$

$$\frac{dx}{dy} - x = y^2$$

$$\mu(y) = e^{\int -1\,dy} = e^{-y}$$

$$e^{-y}x = \int y^2 e^{-y}\,dy = -y^2e^{-y} - 2ye^{-y} - 2e^{-y} + C$$

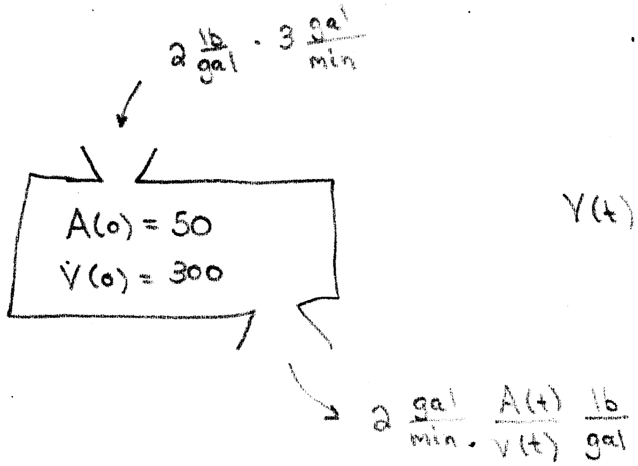
SIGNS	DERIVS	ANTIS
+	y^2	e^{-y}
−	$2y$	$-e^{-y}$
+	2	e^{-y}
−	0	$-e^{-y}$

$$x(y) = -y^2 - 2y - 2 + Ce^{y}$$

$$x(0) = -2 \Rightarrow C = 0$$

$$x(y) = -y^2 - 2y - 2$$

5)



$V(t) = 300 + t$

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}, \quad A(0) = 50$$

$$\frac{dA}{dt} + \frac{2}{300+t} A = 6$$

$$\mu(t) = e^{\int \frac{2}{300+t}\,dt} = e^{2\ln|300+t|}$$
$$= (300+t)^2$$

$$(300+t)^2 A = \int 6(300+t)^2\,dt$$
$$= 2(300+t)^3 + C$$

$$A(t) = 2(300+t) + \frac{C}{(300+t)^2}$$

$$A(0) = 50 \Rightarrow 2(300) + \frac{C}{(300)^2}$$
$$\Rightarrow 50 = 600 + \frac{C}{(300)^2}$$
$$C = -550(300)^2$$

$$\boxed{A(t) = 2(300+t) - \frac{550(300)^2}{(300+t)^2}}$$

WHEN $V = 320$, $t = 20$

$$A(20) = 2(320) - \frac{550(300)^2}{(320)^2}$$
$$\approx 156.6 \text{ lb}$$

6) $m \dfrac{dv}{dt} = -mg - bv, \quad v(0) = v_0$

$$\frac{dv}{dt} + \frac{b}{m} v = -g$$

$$\mu(t) = e^{\int \frac{b}{m} dt} = e^{bt/m}$$

$$e^{bt/m} v(t) = \int -g e^{bt/m} \, dt$$

$$= -\frac{gm}{b} e^{bt/m} + C$$

$$v(t) = -\frac{gm}{b} + C e^{-bt/m}$$

$$v(0) = v_0 \Rightarrow v_0 = -\frac{gm}{b} + C$$

$$\Rightarrow C = v_0 + \frac{gm}{b}$$

$$\boxed{v(t) = -\frac{gm}{b} + \left(v_0 + \frac{gm}{b}\right) e^{-bt/m}}$$

$\lim_{t \to \infty} v(t) = \dfrac{-gm}{b}$, THIS IS THE OBJECT'S TERMINAL VELOCITY.

7) $(\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$ EQUATION IS EXACT.

$$F_x(x,y) = \cos x \sin x - xy^2 \Rightarrow F(x,y) = \frac{1}{2}\sin^2 x - \frac{1}{2}x^2y^2 + g(y)$$

$$\int \cos x \sin x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$\int u \, du = \frac{1}{2}u^2$$

$$F_y(x,y) = y - x^2y \Rightarrow F(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2y^2 + h(x)$$

$$F(x,y) = \boxed{\frac{1}{2}y^2 + \frac{1}{2}\sin^2 x - \frac{1}{2}x^2y^2 = C}$$

8) $(x + \tan^{-1} y)\,dx + \left(\frac{x+y}{1+y^2}\right)dx = 0$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} = \frac{\partial N}{\partial x}$$ EQUATION IS EXACT.

$$F_x(x,y) = x + \tan^{-1} y \Rightarrow F(x,y) = \frac{1}{2}x^2 + x\tan^{-1} y + g(y)$$

$$F_y(x,y) = \frac{x}{1+y^2} + \frac{y}{1+y^2} \Rightarrow F(x,y) = x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) + h(x)$$

$$\int \frac{y}{1+y^2}\,dy = \frac{1}{2}\ln(1+y^2)$$

$u = 1+y^2$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$$F(x,y) = \boxed{\frac{1}{2}x^2 + x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) = C}$$

9) $-y^2\,dx + (x^2+xy)\,dy = 0$

$$\frac{\partial M}{\partial y} = -2y \neq \frac{\partial N}{\partial x} = 2x+y$$ EQUATION IS NOT EXACT.

MULTIPLY BY $(x^2y)^{-1}$.

$$\frac{1}{x^2y}(-y^2).\,dx + \frac{x^2+xy}{x^2y}\,dy = 0.$$

$$\frac{-y}{x^2}\,dx + \frac{x+y}{xy}\,dy = 0$$

$$\frac{-y}{x^2}\,dx + \left(\frac{1}{y}+\frac{1}{x}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}$$ NEW EQUATION IS EXACT.

$$F_x(x,y) = -\frac{y}{x^2} \Rightarrow F(x,y) = \frac{y}{x} + g(y)$$

$$F_y(x,y) = \frac{1}{y}+\frac{1}{x} \Rightarrow F(x,y) = \ln|y| + \frac{y}{x} + h(x)$$

$$F(x,y) = \boxed{\ln|y| + \frac{y}{x} = C}$$