MTH 240 - Assignment #4 Key

1). $\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$

$\frac{dy}{dx} - \frac{2}{x}y = -x^2y^2$

$y^{-2}\frac{dy}{dx} - \frac{2}{x}y^{-1} = -x^2$

$u = y^{-1}$

$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{du}{dx} - \frac{2}{x}u = -x^2$

$\frac{du}{dx} + \frac{2}{x}u = x^2$

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$

$x^2 u(x) = \int x^4\,dx = \frac{1}{5}x^5 + C$

$u(x) = \frac{1}{5}x^3 + \frac{C}{x^2}$

$y(x) = \frac{1}{\frac{1}{5}x^3 + \frac{C}{x^2}} = \frac{5x^2}{x^5 + C}$

$$\boxed{y(x) = \frac{5x^2}{x^5 + C}}$$

2) $(y^2 - xy)\,dx + x^2\,dy = 0$

$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$

$u = \frac{y}{x}$ $\quad ux = y$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = u - u^2$

$x\frac{du}{dx} = -u^2$

$\frac{1}{u^2}du = -\frac{1}{x}dx$

$\int u^{-2}du = -\int \frac{1}{x}dx$

$-\frac{1}{u} = -\ln|x| + C$

$\frac{1}{u} = \ln|x| + C$

$\frac{x}{y} = \ln|x| + C$

$$\boxed{y(x) = \frac{x}{\ln|x| + C}}$$

3) $\dfrac{dy}{dx} = \dfrac{y+x}{x-y} = \dfrac{\frac{y}{x}+1}{1-\frac{y}{x}}$, $y(1)=0$

$u = \dfrac{y}{x}$ $\quad y = ux \quad \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$u + x\dfrac{du}{dx} = \dfrac{u+1}{1-u} \Rightarrow x.\dfrac{du}{dx} = \dfrac{u+1}{1-u} - \dfrac{u(1-u)}{1-u} = \dfrac{u^2+1}{1-u}$

$\dfrac{1-u}{1+u^2}\,du = \dfrac{1}{x}\,dx \Rightarrow \displaystyle\int \frac{1}{1+u^2}\,du - \int \frac{u}{1+u^2}\,du = \int \frac{1}{x}\,dx$

$w = 1+u^2$

$dw = 2u\,du$

$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$

$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C$

$y(1) = 0 \Rightarrow C = 0$

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) - \ln|x| = 0}$$

4) $\dfrac{dy}{dx} = (x+y+2)^2$, $y(0) = -1$

$u = x+y+2$

$\dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$

$\dfrac{du}{dx} - 1 = u^2$

$\dfrac{du}{dx} = u^2+1$

$\dfrac{du}{u^2+1} = dx \Rightarrow \tan^{-1}(x+y+2) = x + C$

$y(x) = \tan(x+C) - 2 - x$

$y(0) = -1 \Rightarrow \tan(C) - 2 = -1$

$C = \frac{\pi}{4}$

$$\boxed{y(x) = \tan\left(x+\frac{\pi}{4}\right) - 2 - x}$$

5) $xy'' + y' = 4x$

$u = y', \quad u' = y''$

$xu' + u = 4x$

$u' + \frac{1}{x}u = 4$

$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$

$= x$ ASSUMING $x > 0$

$x\,u(x) = \int 4x\,dx$

$= 2x^2 + C$

$u(x) = 2x + \frac{C}{x}$

$y'(x) = 2x + \frac{C}{x}$

$$\boxed{y(x) = x^2 + C\ln x + D, \quad x > 0}$$

6) $y'' + y = 0$

$u = y' \quad \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} \Rightarrow y'' = u\frac{du}{dy}$

$$u\frac{du}{dy} + y = 0$$

$$u\,du = -y\,dy$$

$$\frac{u^2}{2} = -\frac{y^2}{2} + C, \quad C > 0$$

$$u^2 = C - y^2$$

$$u = \pm\sqrt{C - y^2}$$

$$\frac{dy}{\pm\sqrt{C-y^2}} = dx$$

$$\frac{\pm 1}{\sqrt{C}} \frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{C}}\right)^2}}\,dy = dx$$

$$u = \frac{y}{\sqrt{C}}$$

$$du = \frac{1}{\sqrt{C}}\,dy$$

$$\pm\int \frac{1}{\sqrt{1-u^2}}\,du = \int dx$$

$$\pm \sin^{-1} u = x + D$$

$$\frac{y}{\sqrt{C}} = \pm \sin(x + D)$$

$$y(x) = \pm\sqrt{C}\sin(x+D)$$

$$= \pm\sqrt{C}\sin x \cos D \pm \sqrt{C}\sin D \cos x$$

$$\boxed{y(x) = A\sin x + B\cos x}$$

7) $xy'' + 5y' = 0, \quad x > 0$

a) $u = y', \quad u' = y''$

$$xu' + 5u = 0$$

$$\frac{1}{u}\,du = -\frac{5}{x}\,dx$$

$$\ln|u| = -5\ln|x| + C$$

$$u = e^{-5\ln|x| + C} = \frac{c}{x^5}, \quad x > 0$$

$$\frac{dy}{dx} = \frac{c}{x^5} \Rightarrow y(x) = -\frac{c}{4}x^{-4} + D$$

$$\boxed{y(x) = \frac{c}{x^4} + D}$$

b) VERIFY $y_1(x) \equiv 1 \ldots$

$y_1(x) = 1$

$y_1'(x) = 0 \qquad xy_1'' + 5y' = x(0) + 5(0) = 0 \ \checkmark$

$y_1''(x) = 0$

VERIFY $y_2(x) = x^{-4} \ldots$

$y_2(x) = x^{-4}, \quad x > 0$

$y_2'(x) = 4x^{-5} \qquad xy_2'' + 5y_2' = x(-20x^{-6}) + 5(4x^{-5})$

$y_2''(x) = -20x^{-6} \qquad = -20x^{-5} + 20x^{-5} = 0 \ \checkmark$

7) CONTINUED...

c)

$$W[y_1, y_2](x) = \begin{vmatrix} 1 & x^{-4} \\ 0 & -4x^{-5} \end{vmatrix} = -\frac{4}{x^5} \neq 0 \text{ FOR ANY } x.$$

$\Downarrow$

SOLUTIONS y_1 AND y_2 ARE LINEARLY INDEP WHEN $x \neq 0$.

d)

$$y(x) = c_1 + \frac{c_2}{x^4}$$

$$y'(x) = -\frac{4c_2}{x^5}$$

$$y(1) = 2 \Rightarrow c_1 + c_2 = 2$$

$$y'(1) = 4 \Rightarrow -4c_2 = 4$$

$$c_2 = -1$$
$$c_1 = 3$$

$$y(x) = 3 - \frac{1}{x^4}$$

8) $yy'' + (y')^2 = 0$

a) VERIFY $y_1(x) \equiv 1 \ldots$

$y_1(x) = 1$

$y_1'(x) = 0$ $\qquad 1(0) + (0)^2 = 0$ ✓

$y_1''(x) = 0$

VERIFY $y_2(x) = \sqrt{x} \ldots$

$y_2(x) = \sqrt{x}$

$y_2'(x) = \dfrac{1}{2\sqrt{x}}$ $\qquad \sqrt{x}\left(-\dfrac{1}{4\sqrt{x^3}}\right) + \left(\dfrac{1}{2\sqrt{x}}\right)^2$

$y_2''(x) = \dfrac{-1}{4\sqrt{x^3}}$ $\qquad = -\dfrac{1}{4x} + \dfrac{1}{4x} = 0$ ✓

b) $y_1(x) + y_2(x) = 1 + \sqrt{x}$

$[y_1(x) + y_2(x)]' = \dfrac{1}{2\sqrt{x}}$

$[y_1(x) + y_2(x)]'' = -\dfrac{1}{4\sqrt{x^3}}$

$(y_1 + y_2)(x)\left[(y_1+y_2)(x)\right]'' + \left[(y_1+y_2)'(x)\right]^2$

$(1+\sqrt{x})\left(-\dfrac{1}{4\sqrt{x^3}}\right) + \dfrac{1}{4x} = -\dfrac{1}{4\sqrt{x^3}} \neq 0$ ✓

c) THIS EQUATION IS NONLINEAR!

I ONLY EXPECT A SUM OF SOLUTIONS TO BE A SOLUTION FOR LINEAR EQUATIONS.

9) $y'' - y' - 12y = 0, \quad y(0) = 2, \quad y'(0) = -5$

CHAR EQN:

$r^2 - r - 12 = 0$

$(r-4)(r+3) = 0$

$r = 4, \ r = -3$

$y(x) = c_1 e^{4x} + c_2 e^{-3x}$

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = -5 \Rightarrow 4c_1 - 3c_2 = -5$

$7c_1 = 1$

$c_1 = \frac{1}{7}$

$c_2 = \frac{13}{7}$

$$\boxed{y(x) = \frac{1}{7} e^{4x} + \frac{13}{7} e^{-3x}}$$

10) $y^{(4)} - y'' = 0$

CHAR EQN:

$r^4 - r^2 = 0$

$r^2 (r^2 - 1) = 0$

$r^2 (r+1)(r-1) = 0$

$r = 0, \ r = 0, \ r = -1, \ r = 1$

$y(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{-x} + c_4 e^{x}$

$$\boxed{y(x) = c_1 + c_2 x + c_3 e^{-x} + c_4 e^{x}}$$