MTH 240 Assignment #6 key

1) $2x'' + x' + 16x = 0; \quad x(0) = -1, \; x'(0) = 4$

Char equation: $2r^2 + r + 16 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4(2)(16)}}{2(2)} = \frac{-1 \pm \sqrt{127}\, i}{4}$$

$$\alpha = -\frac{1}{4}$$
$$\beta = \frac{\sqrt{127}}{4}$$

$$x(t) = c_1 e^{-t/4} \cos\left(\frac{\sqrt{127}}{4} t\right) + c_2 e^{-t/4} \sin\left(\frac{\sqrt{127}}{4} t\right)$$

$x(0) = -1 \Rightarrow c_1 = -1$

$x'(0) = 4 \Rightarrow -\frac{c_1}{4} + \frac{\sqrt{127}}{4} c_2 = 4$

$$c_2 = \frac{15}{\sqrt{127}}$$

$$A = \sqrt{1 + \frac{225}{127}} = \sqrt{\frac{352}{127}}$$

$\tan \varphi = \dfrac{-1}{15/\sqrt{127}}$, φ in Quad IV $\Rightarrow \varphi = \tan^{-1}\left(-\frac{\sqrt{127}}{15}\right)$

$$\boxed{x(t) = \sqrt{\frac{352}{127}}\, e^{-t/4} \sin\left(\frac{\sqrt{127}}{4} t + \tan^{-1}\left(-\frac{\sqrt{127}}{15}\right)\right)}$$

2) $9x'' + 6x' + 37x = 0; \quad x(0) = 1, \; x'(0) = -2$

CHAR. EQUATION : $9r^2 + 6r + 37 = 0$

$$(3r+1)^2 = -36$$

$$r = \frac{-1 \pm 6i}{3} \quad . \quad \alpha = -\frac{1}{3}, \; \beta = 2$$

$$x(t) = c_1 e^{-t/3} \cos 2t + c_2 e^{-t/3} \sin 2t$$

$$x(0) = 1 \Rightarrow c_1 = 1$$

$$x'(0) = -2 = -\frac{c_1}{3} + 2c_2 = -2$$

$$2c_2 = -\frac{5}{3}$$

$$c_2 = \frac{-5}{6}$$

$$A = \sqrt{1 + \frac{25}{36}} = \sqrt{\frac{61}{36}} = \frac{\sqrt{61}}{6}$$

$\tan \varphi = \dfrac{1}{-5/6}$ AND φ IN QUAD II

$$\varphi = \tan^{-1}\left(-\frac{6}{5}\right) + \pi$$

$$\boxed{x(t) = \frac{\sqrt{61}}{6} e^{-t/3} \sin\left(2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right)\right)}$$

PASSES THROUGH EQUILIBRIUM WHEN $2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right) = k\pi$

OR $t = \dfrac{(k-1)\pi - \tan^{-1}\left(-\frac{6}{5}\right)}{2}$

$k=1$ GIVES 1ST TIME THROUGH eq.

3RD TIME $\Rightarrow k = 3 \Rightarrow \boxed{t \approx 3.58 \text{ SEC}}$

3) $y'' - y' + 9y = 3\sin 2t$

Homo. Eqn: $y''-y'+9y=0$

Char. Eqn: $r^2-r+9=0$

$$r = \frac{1 \pm \sqrt{1-36}}{2} = \frac{1 \pm \sqrt{35}\,i}{2}$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{\sqrt{35}}{2}$$

$$y_c(t) = c_1 e^{t/2} \cos\left(\frac{\sqrt{35}}{2}t\right) + c_2 e^{t/2} \sin\left(\frac{\sqrt{35}}{2}t\right)$$

Non. Homo. Eqn has $g(t) = 3\sin 2t$.

$$y_p(t) = t^s (A\cos 2t + B\sin 2t)$$

Choose $S = 0$.

$$y_p(t) = A\cos 2t + B\sin 2t$$

$$y_p'(t) = -2A\sin 2t + 2B\cos 2t$$

$$y_p''(t) = -4A\cos 2t - 4B\sin 2t$$

$$\begin{aligned} y_p'' - y_p' + 9y_p &= -4A\cos 2t - 4B\sin 2t \\ &\quad - 2B\cos 2t + 2A\sin 2t \\ &\quad + 9A\cos 2t + 9B\sin 2t \\ &= (5A-2B)\cos 2t + (5B+2A)\sin 2t \\ &= 3\sin 2t \end{aligned}$$

$$\begin{aligned} 5A - 2B &= 0 \\ 5B + 2A &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 10A - 4B &= 0 \\ -25B - 10A &= -15 \\ \hline -29B &= -15 \end{aligned}$$

$$B = \frac{15}{29}$$

$$A = \frac{6}{29}$$

$$y_p(t) = \frac{6}{29}\cos 2t + \frac{15}{29}\sin 2t$$

$$y(t) = c_1 e^{t/2} \cos\left(\frac{\sqrt{35}}{2}t\right) + c_2 e^{t/2} \sin\left(\frac{\sqrt{35}}{2}t\right) + \frac{6}{29}\cos(2t) + \frac{15}{29}\sin(2t)$$

4) $y'' - 8y' + 7y = x + 3e^{7x}; \quad y(0)=1,\ y'(0)=1$

Homo. eqn: $y'' - 8y' + 7y = 0$

Char. eqn: $r^2 - 8r + 7 = 0$

$$(r-7)(r-1) = 0$$

$$r = 7,\ r = 1$$

$$y_c(x) = c_1 e^{7x} + c_2 e^{x}$$

Nonhomo. eqn. has $g(x) = x + 3e^{7x}$

$$y_p(x) = x^r(Ax+B) + x^s C e^{7x}$$

Choose $r = 0,\ s = 1$

$$y_p(x) = Ax + B + Cxe^{7x}$$

$$y_p'(x) = A + Ce^{7x} + 7Cxe^{7x}$$

$$y_p''(x) = 14Ce^{7x} + 49Cxe^{7x}$$

$$\begin{aligned} y_p'' - 8y_p' + 7y_p &= 14Ce^{7x} + 49Cxe^{7x} \\ &\quad - 8A - 8Ce^{7x} - 56Cxe^{7x} \\ &\quad + 7Ax + 7B + 7Cxe^{7x} \\ &= 7Ax + (7B - 8A) + 6Ce^{7x} \\ &= x + 3e^{7x} \end{aligned}$$

$$\Downarrow$$

$7A = 1 \qquad A = \frac{1}{7}$

$7B - 8A = 0 \qquad B = \frac{8}{49}$

$6C = 3 \qquad C = \frac{1}{2}$

$$y_p(x) = \frac{1}{7}x + \frac{8}{49} + \frac{1}{2}xe^{7x}$$

$$y(x) = c_1 e^{7x} + c_2 e^{x} + \frac{1}{7}x + \frac{8}{49} + \frac{1}{2}xe^{7x}$$

$y(0) = 1 \Rightarrow c_1 + c_2 + \frac{8}{49} = 1$

$y'(0) = 1 \Rightarrow 7c_1 + c_2 + \frac{1}{7} + \frac{1}{2} = 1$

$$\begin{aligned} c_1 + c_2 &= \frac{41}{49} \\ 7c_1 + c_2 &= \frac{5}{14} \\ \hline 6c_1 &= -\frac{47}{98} \end{aligned} \Rightarrow c_1 = \frac{-47}{588}$$

$$c_2 = \frac{11}{12}$$

$$\boxed{y(x) = \frac{11}{12}e^{x} - \frac{47}{588}e^{7x} + \frac{1}{7}x + \frac{8}{49} + \frac{1}{2}xe^{7x}}$$

5) $y'' - 6y' + 9y = 5t^6 e^{3t}$

Homo. EQN: $y'' - 6y' + 9y = 0$

CHAR EQN: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3,\ r = 3$

$$\boxed{y_c(t) = C_1 e^{3t} + C_2 t e^{3t}}$$

NON HOMO. EQN. HAS $g(t) = 5t^6 e^{3t}$

$y_p(t) = t^s \left(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G\right) e^{3t}$

MUST CHOOSE $S = 2$

THEREFORE, $\boxed{y_p(t) = \left(At^8 + Bt^7 + Ct^6 + Dt^5 + Et^4 + Ft^3 + Gt^2\right) e^{3t}}$

IT TURNS OUT THAT $y(t) = C_1 e^{3t} + C_2 t e^{3t} + \frac{5}{56} t^8 e^{3t}$.

6) $y'' + y = \sin t + t\cos t + e^t$

Homo. eqn: $y'' + y = 0$

Char. eqn: $r^2 + 1 = 0$

$r = \pm i$

$\alpha = 0, \ \beta = 1$

$$\boxed{y_c(t) = c_1 \cos t + c_2 \sin t}$$

NonHomo. eqn has $g(t) = \sin t + t\cos t + e^t$

For $g_1(t) = \sin t$, we need

$y_{p_1}(t) = t^s(A\sin t + B\cos t)$, where $s = 1$

For $g_2(t) = t\cos t$, we need

$y_{p_2}(t) = t^s\left[(Ct + D)\sin t + (Et + F)\cos t\right]$, where $s = 1$

For $g_3(t) = e^t$, we need

$y_{p_3}(t) = t^s(Ge^t)$, where $s = 0$.

All together, this comes to

$$\boxed{y_p(t) = (At^2 + Bt)\sin t + (Ct^2 + Dt)\cos t + Ee^t}$$

It turns out that, $y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{4} t\cos t + \frac{1}{4} t^2 \sin t + \frac{1}{2} e^t$

7) $y^{(3)} + y'' = x + e^{-x}$; $y(0)=1$, $y'(0)=0$, $y''(0)=1$

Homo eqn: $y^{(3)} + y'' = 0$

Char. eqn: $r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r=0,\ r=0,\ r=-1$

$y_c(x) = c_1 + c_2 x + c_3 e^{-x}$

NonHomo eqn has $g(x) = x + e^{-x}$

For $g_1(x) = x$, we need $y_{p_1}(x) = x^s(Ax+B)$, where $s=2$

For $g_2(x) = e^{-x}$, we need $y_{p_2}(x) = x^s(Ce^{-x})$, where $s=1$

$y_p(x) = Ax^3 + Bx^2 + Cxe^{-x}$

$y_p'(x) = 3Ax^2 + 2Bx + Ce^{-x} - Cxe^{-x}$

$y_p''(x) = 6Ax + 2B - Ce^{-x} - Ce^{-x} + Cxe^{-x}$

$y_p^{(3)}(x) = 6A + Ce^{-x} + Ce^{-x} + Ce^{-x} - Cxe^{-x}$

$y_p^{(3)} + y_p'' = (6A+2B) + 6Ax + Ce^{-x} = x + e^{-x}$

$6A + 2B = 0 \Rightarrow B = -\frac{1}{2}$

$6A = 1 \Rightarrow A = \frac{1}{6}$

$C = 1$

$y_p(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$

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$$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$$

$$y(0) = 1 \Rightarrow c_1 + c_3 = 1$$

$$y'(x) = c_2 - c_3 e^{-x} + \frac{1}{2}x^2 - x + e^{-x} - xe^{-x}$$

$$y'(0) = 0 \Rightarrow c_2 - c_3 + 1 = 0$$

$$y''(x) = c_3 e^{-x} + x - 1 - e^{-x} - e^{-x} + xe^{-x}$$

$$y''(0) = 1 \Rightarrow c_3 - 1 - 1 - 1 = 1$$

$$c_3 = 4$$

$$c_2 = 3$$

$$c_1 = -3$$

$$\boxed{y(x) = -3 + 3x + 4e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}}$$