MTH 240 Assignment #7 Key

1) $y'' - 2y' - 8y = 3e^{-2x}$

Homo. Eqn: $y'' - 2y' - 8y = 0$

Char. Eqn: $r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$r = 4,\ r = -2$

$y_c(x) = c_1 e^{4x} + c_2 e^{-2x}$

Variation of parameters...

$g(x) = 3e^{-2x}$

$$W = \begin{vmatrix} e^{4x} & e^{-2x} \\ 4e^{4x} & -2e^{-2x} \end{vmatrix} = -6e^{2x}$$

$$V_1(x) = \int \frac{-3e^{-2x}e^{-2x}}{-6e^{2x}}\,dx = \int \frac{1}{2} e^{-6x}\,dx$$

$$= -\frac{1}{12} e^{-6x}$$

$$V_2(x) = \int \frac{3e^{-2x}e^{4x}}{-6e^{2x}}\,dx = \int -\frac{1}{2}\,dx = -\frac{1}{2}x$$

$y_p(x) =$
$V_1(x)y_1(x) + V_2(x)y_2(x) = -\frac{1}{12}e^{-2x} - \frac{1}{2}xe^{-2x}$

$$\boxed{y(x) = c_1 e^{4x} + c_2 e^{-2x} - \frac{1}{2}xe^{-2x}}$$

2) $x'' - 4x' + 4x = \dfrac{e^{2t}}{t^2}, \quad t > 0$

HOMO. EQN: $x'' - 4x' + 4x = 0$

CHAR. EQN: $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0.$$

$$r = 2,\ r = 2$$

$$x_c(t) = c_1 e^{2t} + c_2 t e^{2t}$$

VARIATION OF PARAMETERS...

$$g(t) = \frac{e^{2t}}{t^2}$$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{4t}$$

$$v_1(t) = \int -\frac{\frac{e^{2t}}{t^2}\, te^{2t}}{e^{4t}}\, dt = \int -\frac{1}{t}\, dt$$

$$= -\ln t$$

$$v_2(t) = \int \frac{\frac{e^{2t}}{t^2}\, e^{2t}}{e^{4t}}\, dt = \int t^{-2}\, dt$$

$$= -\frac{1}{t}$$

$$x_p(t) = v_1(t)\, x_1(t) + v_2(t)\, x_2(t)$$

$$= -(\ln t)\, e^{2t} - \frac{1}{t}(te^{2t})$$

$$x(t) = c_1 e^{2t} + c_2 t e^{2t} - (\ln t)\, e^{2t}$$

3) $y'' + y = \csc^2 x$

Homo. EQN: $y'' + y = 0$

CHAR EQN: $r^2 + 1 = 0$

$r = \pm i$

$y_c(x) = C_1 \cos x + C_2 \sin x$

VARIATION OF PARAMETERS...

$g(x) = \csc^2 x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$V_1(x) = \int -\csc^2 x \sin x \, dx = \int -\csc x \, dx$$

$$= \ln|\csc x + \cot x|$$

$$V_2(x) = \int \csc^2 x \cos x \, dx$$

$$= \int \cot x \csc x \, dx$$

$$= -\csc x$$

$$y_p(x) = V_1(x) y_1(x) + V_2(x) y_2(x)$$

$$= (\cos x) \ln|\csc x + \cot x| - 1$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x + (\cos x) \ln|\csc x + \cot x| - 1}$$

4) $(1-x)y'' + xy' - y = (1-x)^2, \quad x > 1$

a) $y_1(x) = x$ $\qquad (1-x)(0) + x(1) - x = 0$ ✓

$y_1'(x) = 1$

$y_1''(x) = 0$

$y_2(x) = e^x$ $\qquad (1-x)e^x + xe^x - e^x = 0$ ✓

$y_2'(x) = e^x$

$y_2'(x) = e^x$

y_1 & y_2 ARE SOLUTIONS OF THE CORRESPONDING HOMOGENEOUS EQUATION.

b) $W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1) \neq 0$ WHEN $x \neq 1$

y_1 & y_2 ARE LIN. INDEP. WHEN $x > 1$.

c) $g(x) = \dfrac{(1-x)^2}{1-x} = 1-x$

$$V_1(x) = \int \frac{(x-1)e^x}{e^x(x-1)}\,dx = \int 1\,dx = x$$

$$V_2(x) = \int \frac{(1-x)x}{e^x(x-1)}\,dx = \int -xe^{-x}\,dx = xe^{-x} - \int e^{-x}dx = xe^{-x} + e^{-x}$$

$u = -x \qquad du = -dx$

$dv = e^{-x}dx \qquad V = -e^{-x}$

CONTINUED →

$$y_p(x) = x^2 + (xe^{-x} + e^{-x})\, e^x$$

$$= x^2 + x + 1$$

$$\boxed{y_p(x) = x^2 + x + 1}$$

d) $\boxed{y(x) = c_1 x + c_2 e^x + x^2 + x + 1}$

e) $y(2) = 1 \Rightarrow 2c_1 + c_2 e^2 + 7 = 1$

$$y'(x) = c_1 + c_2 e^x + 2x + 1$$

$$y'(2) = -4 \Rightarrow c_1 + c_2 e^2 + 5 = -4$$

$$2c_1 + e^2 c_2 = -6$$
$$c_1 + e^2 c_2 = -9$$

$$c_1 = 3$$

$$c_2 = \frac{-12}{e^2}$$

$$\boxed{y(x) = 3x - \frac{12}{e^2} e^x + x^2 + x + 1}$$