Math 240 - Test 1

September 11, 2025

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

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1. (4 points) Consider the differential equation $\frac{dy}{dx} = 2xy$.

   (a) Steve "solved" the equation as follows:

   $$dy = 2xy\,dx \quad \Longrightarrow \quad \int dy = \int 2xy\,dx \quad \Longrightarrow \quad y = x^2y + C,$$

   and then he solved for y. Explain why Steve is all wrong.

   HE TREATED y, ON THE RIGHT, AS A KNOWN INDEPENDENT VARIABLE. IT IS NOT!

   (b) Without solving the equation, explain how Steve should have solved it.

   COULD HAVE SEPARATED VARIABLES: $\frac{1}{y}dy = 2x\,dx \Rightarrow \int \frac{1}{y}dy = \int 2x\,dx$

   THIS IS GOOD!

2. (6 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $(1-x)y'' - 4xy' + 5y = \cos x$

   ORDINARY, LINEAR, 2^{ND} ORDER

   (b) $x\frac{d^3y}{dx^3} - 2\left(\frac{dy}{dx}\right)^4 + y = 0$

   ORDINARY, NONLINEAR, 3^{RD} ORDER

   (c) $10t\frac{\partial^2 u}{\partial x^2} = x\frac{\partial u}{\partial t}$

   PARTIAL, LINEAR, 2^{ND} ORDER

   (d) $x^2\frac{d^2x}{dt^2} + 5x\frac{dx}{dt} + 6t = 1$

   ORDINARY, NONLINEAR, 2^{ND} ORDER

3. (5 points) The equation $y'' - y = 0$ has the general solution $y(x) = c_1 e^x + c_2 e^{-x}$, where c_1 and c_2 are arbitrary constants. Find the particular solution that satisfies the initial conditions $y(0) = 3$ and $y'(0) = 8$.

$y(0) = 3 \Rightarrow C_1 + C_2 = 3$

$y'(x) = C_1 e^x - C_2 e^{-x}$

$y'(0) = 8 \Rightarrow C_1 - C_2 = 8$

$$\begin{array}{r} C_1 + C_2 = 3 \\ C_1 - C_2 = 8 \\ \hline 2C_1 = 11 \end{array}$$

$C_1 = \frac{11}{2}$

$C_2 = 3 - \frac{11}{2} = -\frac{5}{2}$

$\boxed{C_1 = \frac{11}{2},\ C_2 = -\frac{5}{2},\ y = \frac{11}{2}e^x - \frac{5}{2}e^{-x}}$

4. (12 points) Analyze each initial value problem to determine which one of these applies. **Do not attempt to solve the equations.**

(A) A solution exists, but it is not guaranteed to be unique.

(B) There is a unique solution.

(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $\left(\frac{dy}{dx}\right)^2 + x^2 = y, \quad y(1) = 1$

$$\frac{dy}{dx} = \pm\sqrt{y - x^2}$$

$f(x,y) = \pm\sqrt{y - x^2}$

NOT CONTINUOUS ON ANY RECT. AROUND $(1,1)$. (THERE WILL BY $y - x^2 < 0$.)

$\boxed{(C)}$

(b) $(x-5)\frac{dy}{dx} + (x^2 - 4)y = |x|, \quad y(0) = 5$

$$\frac{dy}{dx} + \underbrace{\frac{x^2-4}{x-5}}_{p(x)} y = \underbrace{\frac{|x|}{x-5}}_{q(x)},$$

THIS EQUATION IS LINEAR AND p, q ARE CONTINUOUS FOR ALL $x \neq 5$. THERE WILL BE A UNIQUE SOLUTION THROUGH $(0,5)$.

$\boxed{(B)}$

(c) $\frac{dy}{dx} = \frac{\sqrt[3]{x-y}}{x^2+1}, \quad y(4) = 4$

$f(x,y) = \frac{\sqrt[3]{x-y}}{x^2+1}$. f IS CONTINUOUS EVERYWHERE.

$$f_y(x,y) = \frac{-\frac{1}{3}(x-y)^{-2/3}}{x^2+1} = \frac{-1}{3(x^2+1)\sqrt[3]{(x-y)^2}}$$

f_y IS CONTINUOUS EVERYWHERE EXCEPT WHERE $x = y$.

$\boxed{(A)}$

5. (9 points) Consider the equation $\frac{dy}{dx} = (xy - x)^2$. $= x^2(y-1)^2$

(a) Find the general solution.

$\frac{1}{(y-1)^2}\,dy = x^2\,dx$ ← WE'RE ASSUMING $y \neq 1$

$\int \frac{1}{(y-1)^2}\,dy = x^2\,dx$

$u = y-1$
$du = dy$

$-\frac{1}{y-1} = \frac{1}{3}x^3 + C_1$

$\frac{1}{y-1} = C_2 - \frac{1}{3}x^3$

$y - 1 = \frac{1}{C_2 - \frac{1}{3}x^3} = \frac{3}{C_3 - x^3}$

$y(x) = 1 + \frac{3}{C - x^3}$

(b) Show that your general solution cannot satisfy $y(0) = 1$.

$y(0) = 1 \Rightarrow 1 + \frac{3}{C} = 1 \Rightarrow \underbrace{\frac{3}{C} = 0}$

NOT POSSIBLE!
NO SUCH C.

(c) Find a singular solution satisfying $y(0) = 1$.

$\boxed{y(x) \equiv 1}$

THIS CONST. FUNCTION SATISFIES THE EQUATION AND THE CONDITION!

6. (5 points) Recall that Newton's 2nd Law states that the acceleration (dv/dt) of an object is proportional to the sum of forces acting on the object. A falling object is acted on by a downward force equal to its weight, W, and an upward force (air resistance) that is proportional to its velocity. Write a differential equation for the object's velocity.

V IS NEG. SO $-\beta v$ IS AN UPWARD FORCE. ($\beta > 0$)

ASSUMING W IS POSITIVE,

THEN $\boxed{\frac{dv}{dt} = k(-W - \beta v) \text{ WHERE } k, \beta > 0}$

7. (8 points) Use Euler's method with $h = 0.1$ to estimate $y(0.4)$.

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0$$

$f(x,y) = x + y^2$

$y_{n+1} = y_n + 0.1\, f(x_n, y_n)$

$x_{n+1} = x_n + h$

$\Downarrow$

$y_0 = 0, \; x_0 = 0$

$y_1 = 0 + 0.1(0 + 0^2) = 0$

$x_1 = 0.1 \qquad y(0.1) \approx 0$

$y_2 = 0 + 0.1(0.1 + 0^2)$

$= 0.01 \qquad y(0.2) \approx 0.01$

$x_2 = 0.2$

$y_3 = 0.01 + 0.1(0.2 + 0.01^2)$

$= 0.03001$

$x_3 = 0.3 \qquad y(0.3) \approx 0.03001$

$y_4 = 0.03001 + 0.1(0.3 + 0.03001^2)$

≈ 0.0601006

$x_4 = 0.4$

$\boxed{y(0.4) \approx 0.0601}$

8. (12 points) Solve the initial value problem: $\dfrac{dN}{dt} + \dfrac{10N}{2t+3} = 8, \quad N(0) = 0 \quad (t \geq 0)$

$p(t) = \dfrac{10}{2t+3}, \quad q(t) = 8$

$\mu(t) = e^{\int \frac{10}{2t+3}\, dt} = e^{5 \ln(2t+3)}, \quad t \geq 0$

$= (2t+3)^5$

$\mu(t) \cdot N(t) = \int \mu(t)\, q(t)\, dt$

$(2t+3)^5\, N(t) = \int 8(2t+3)^5\, dt$

$= \frac{4}{6}(2t+3)^6 + C$

$N(t) = \frac{2}{3}(2t+3) + \dfrac{C}{(2t+3)^5}$

$N(0) = 0 \Rightarrow 2 + \dfrac{C}{3^5} = 0$

$\Rightarrow C = -2 \cdot 3^5$

$N(t) = \frac{2}{3}(2t+3) - \dfrac{2 \cdot 3^5}{(2t+3)^5}$

$\boxed{N(t) = \frac{2}{3}(2t+3) - \dfrac{486}{(2t+3)^5}}$

9. (12 points) Find a function $N(x,y)$ so that the equation is exact. Then using that $N(x,y)$, find the general solution.

$$\underbrace{(x + y^2 \sin x + x^2 y - xe^{2y})}_{M}\, dx + N(x,y)\, dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \sin x + x^2 - 2xe^{2y} = \frac{\partial N}{\partial x} \Rightarrow N(x,y) = -2y\cos x + \tfrac{1}{3}x^3 - x^2 e^{2y} + \underset{\text{ANY FUNC OF } y}{\cdot}$$

I'LL USE $N(x,y) = -2y\cos x + \frac{1}{3}x^3 - x^2e^{2y}$

$F_x(x,y) = x + y^2 \sin x + x^2y - xe^{2y} \Rightarrow F(x,y) = \frac{1}{2}x^2 - y^2\cos x + \frac{1}{3}x^3y - \frac{1}{2}x^2e^{2y} + g(y)$

$F_y(x,y) = -2y\cos x + \frac{1}{3}x^3 - x^2e^{2y} \Rightarrow F(x,y) = -y^2\cos x + \frac{1}{3}x^3y - \frac{1}{2}x^2e^{2y} + h(x)$

$$F(x,y) = \boxed{-y^2\cos x + \tfrac{1}{3}x^3y - \tfrac{1}{2}x^2e^{2y} + \tfrac{1}{2}x^2 = C}$$

10. (9 points) Show (or explain) that this equation is NOT separable, NOT linear, and NOT exact.

$$\frac{dy}{dx} + x^2y - e^x y^3 = 0$$

NOT SEPARABLE

$$\frac{dy}{dx} = e^x y^3 - x^2 y$$

CANNOT BE WRITTEN $h(x)\,g(y)$.

NOT EXACT

$$(x^2y - e^x y^3)\,dx + dy = 0$$

$$\frac{\partial M}{\partial y} = x^2 - 3e^x y^2 \neq \frac{\partial N}{\partial x} = 0$$

NOT LINEAR

WELL, OBVIOUSLY THE y^3 THROWS LINEARITY OFF!

Follow-up question: Does the equation fit into any type we have studied. If so, which?

$\frac{dy}{dx} + x^2 y = e^x y^3$ IS A BERNOULLI EQUATION WITH $n=3$.

(WE'D USE $u = y^{-2}$ TO SOLVE.)

Intentionally blank.

The following problems are take-home problems. They are due September 16. You must work on your own.

11. (3 points) A tank is initially filled with 1000 gal of a salt solution that has a salt concentration of 0.002 lb/gal. A salt solution containing 0.25 lb of salt per gallon enters the tank at 8 gal/min and is uniformly mixed. The mixed solution leaves the tank at 6 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up the appropriate initial value problem to determine $A(t)$. DO NOT SOLVE.

$$0.25 \frac{\text{lb}}{\text{gal}} \cdot 8 \frac{\text{gal}}{\text{min}} = 2\ \text{lb}/\text{min}$$

$$V(t) = 1000 + 2t$$

$$\boxed{\begin{aligned} V(0) &= 1000 \\ A(0) &= 2 \end{aligned}}$$

$$6 \frac{\text{gal}}{\text{min}} \cdot \frac{A(t)}{V(t)} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} = 2 - \frac{6A}{1000+2t}$$

$$\boxed{\frac{dA}{dt} = 2 - \frac{3A}{500+t}, \quad A(0) = 2}$$

12. (9 points) Solve the initial value problem: $x\dfrac{dy}{dx} - (1+x)y = xy^2, \quad y(1) = 1$

$$\frac{dy}{dx} - \frac{1+x}{x} y = y^2$$

$$y^{-2}\frac{dy}{dx} - \frac{1+x}{x} y^{-1} = 1$$

$$u = y^{-1}$$

$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-\frac{du}{dx} - \frac{1+x}{x} u = 1$$

$$\frac{du}{dx} + \frac{1+x}{x} u = -1$$

$$\mu(x) = e^{\int(\frac{1}{x}+1)\,dx} = e^{\ln|x| + x} = xe^x, \quad x > 0$$

$$xe^x\, u(x) = \int -xe^x\, dx \qquad v = -x \quad dv = -dx \quad dw = e^x dx \quad w = e^x$$

$$= -xe^x + \int e^x dx$$

$$= -xe^x + e^x + C$$

$$u(x) = -1 + \frac{1}{x} + \frac{C}{xe^x}$$

$$y(x) = \frac{1}{-1 + \frac{1}{x} + \frac{C}{xe^x}}$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{\frac{C}{e}} \Rightarrow C = e$$

$$\boxed{y(x) = \frac{1}{-1 + \frac{1}{x} + \frac{e}{xe^x}}}$$

13. (6 points) Suppose that the amount of a substance decreases at a rate that is inversely proportional to the amount present. If 12 units of the substance are present initially and 8 units are present after 2 days, how long will it take the substance to disappear?

$$\frac{dA}{dt} = \frac{k}{A}, \quad k<0, \quad A(0) = 12, \quad A(2) = 8$$

$$A\,dA = k\,dt$$

$$\frac{1}{2}A^2 = kt + C$$

$$A(0) = 12 \Rightarrow C = 72$$

$$A(2) = 8. \Rightarrow 32 = 2k + 72$$

$$k = -20$$

$$\frac{1}{2}A^2 = 72 - 20t$$

$$A^2 = 144 - 40t$$

$$\boxed{A(t) = \sqrt{144-40t}}$$

$$A(t) = 0 \Rightarrow 144 - 40t = 0$$

$$\Downarrow$$

$$t = \frac{144}{40} = \frac{18}{5}$$

$$\boxed{t = 3.6 \text{ DAYS}}$$