Math 240 - Test 3
November 6, 2025

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (16 points) Find the general solution: $y'' + 4y' - 5y = (3+t)e^t$.

Homo. eqn: $y'' + 4y' - 5y = 0$

Char. eqn: $r^2 + 4r - 5 = 0$

$(r+5)(r-1) = 0$

$r = -5,\ r = 1$

$y_c(t) = c_1 e^{-5t} + c_2 e^t$

Non Homo. eqn has $g(t) = (3+t)e^t$

$y_p(t) = t^s(At+B)e^t$

$s = 1$

$y_p(t) = (At^2 + Bt)e^t$

$y_p'(t) = (2At+B)e^t + (At^2+Bt)e^t$

$y_p''(t) = 2Ae^t + 2(2At+B)e^t + (At^2+Bt)e^t$

$y_p'' + 4y_p' - 5y_p =$

$2Ae^t + 2(2At+B)e^t + \cancel{(At^2+Bt)}e^t$

$+ 4(2At+B)e^t + \cancel{4(At^2+Bt)}e^t$

$- \cancel{5(At^2+Bt)}e^t$

$= (3+t)e^t$

$2A + 6B = 3$

$12A = 1 \qquad A = \frac{1}{12}$

$6B = 3 - \frac{1}{6} \Rightarrow B = \frac{17}{36}$

$y_p(t) = \left(\frac{1}{12}t^2 + \frac{17}{36}t\right)e^t$

$$y(t) = c_1 e^{-5t} + c_2 e^t + \left(\frac{1}{12}t^2 + \frac{17}{36}t\right)e^t$$

2. (18 points) For $t > 0$, consider the equation

$$ty'' + (1-2t)y' + (t-1)y = te^t.$$

The functions $y_1(t) = e^t$ and $y_2(t) = e^t \ln t$ are solutions of the corresponding homogeneous equation.

(a) Compute the Wronskian of y_1 and y_2 and confirm that these functions are linearly independent.

$$W = \begin{vmatrix} e^t & e^t \ln t \\ e^t & e^t \ln t + \frac{e^t}{t} \end{vmatrix} = \frac{e^{2t}}{t} > 0. \text{ WHEN } t > 0$$

$\Rightarrow$ y_1 & y_2 LIN. INDEP.

(b) Use variation of parameters to find a particular solution. (Hint: Be sure to divide by t before applying the variation of parameters formulas.)

$g(t) = e^t$

$$V_1(t) = \int \frac{-e^t e^t \ln t}{e^{2t}/t}\, dt = \int -t \ln t \, dt = -\frac{t^2}{2}\ln t + \int \frac{t}{2}\, dt$$

$u = \ln t \quad du = \frac{1}{t}dt$

$dv = -t \quad v = -\frac{t^2}{2}$

$$= -\frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$V_2(t) = \int \frac{e^t e^t}{e^{2t}/t}\, dt = \int t \, dt = \frac{t^2}{2}$$

$$y_p(t) = V_1(t)\,y_1(t) + V_2(t)\,y_2(t)$$

$$= -\frac{t^2}{2}e^t \ln t + \frac{t^2}{4}e^t + \frac{t^2}{2}e^t \ln t = \frac{t^2}{4}e^t$$

$$\boxed{y_p(t) = \frac{t^2}{4}e^t}$$

(c) What is the general solution?

$$\boxed{y(t) = c_1 e^t + c_2 e^t \ln t + \frac{t^2}{4} e^t}$$

3. (3 points) Determine all singular points of the equation $xy'' + \frac{e^x}{x+5}y' + (\cos x)y = 0$.

$$y'' + \frac{e^x}{x(x+5)}y' + \frac{\cos x}{x}y = 0$$

SINGULAR PTS : $x(x+5) = 0$

$$\boxed{x = 0, \quad x = -5}$$

4. (9 points) Find a minimum value for the radius of convergence of a power series solution centered at $x = c$.

(a) $(x+1)y'' - 3xy' + 2y = 0, \qquad c = 1$

$$y'' - \frac{3x}{x+1}y' + \frac{2}{x+1}y = 0$$

SING. PT. $x = -1$

DISTANCE FROM SING. PT. TO CENTER TO 2 UNITS. $\boxed{R \geq 2}$

(b) $y'' - xy' - 3y = 0, \qquad c = 2$

THERE ARE NO SINGULAR PTS.

$$\boxed{R = \infty}$$

(c) $\underbrace{(x^2 - 5x + 6)}_{(x-3)(x-2)}y'' - 3xy' - y = 0, \qquad c = 0$

$$y'' - \frac{3x}{(x-3)(x-2)}y' - \frac{1}{(x-3)(x-2)}y = 0$$

SING. PTS. $x = 3$, $x = 2$

NEAREST TO CENTER

DISTANCE FROM 2 TO 0 IS 2 UNITS.

$$\boxed{R \geq 2}$$

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

5. (19 points) Consider the equation $(x^2+1)y''-xy'+y=0$. Notice that a power series solution centered at $x=0$ has radius of convergence of at least 1.

(a) Find the complete recurrence relation for the power series solution centered at $x=0$.

$$0=\sum_{n=2}^{\infty} n(n-1)a_n x^n+\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n$$

$n \leftrightarrow n+2$

$$=\sum_{n=2}^{\infty} n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=0}^{\infty} n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n.$$

$$=\sum_{n=0}^{\infty}\Big[\underbrace{n(n-1)a_n+(n+2)(n+1)a_{n+2}-(n-1)a_n}_{}\Big]x^n$$

$$(n-1)^2 a_n+(n+2)(n+1)a_{n+2}=0$$

$$a_{n+2}=\frac{-(n-1)^2}{(n+2)(n+1)}a_n; \quad n=0,1,2,3,\ldots$$

a_0 ARB,

a_1 ARB.

(b) One of the two linearly independent solutions is a very simple polynomial. Find it. Then find the first four nonzero terms of the second ~~linearly independent solution.~~

a_0 ARB	a_1 ARB
$a_2=\frac{-1}{2}a_0$	$a_3=0$
$a_4=\frac{1}{4!}a_0$	$a_5=0$
$a_6=\frac{-9}{6!}a_0$	$\vdots$

$$\boxed{\begin{aligned} y_1(x)&=a_0-\tfrac{1}{2}a_0x^2+\tfrac{1}{24}a_0x^4-\tfrac{9}{720}a_0x^6+\cdots \\ y_2(x)&=a_1x \end{aligned}}$$

(c) Find the solution that satisfies $y(0)=0$ and $y'(0)=5$.

$y(x)=y_1(x)+y_2(x)$

$y(0)=0 \Rightarrow a_0=0$

$y'(0)=5 \Rightarrow a_1=5$

$$\boxed{y(x)=5x}$$

6. (10 points) Find the complete recurrence relation for the power series solution centered at $x = 0$.

$$y'' - xy' - x^2 y = 0$$

$y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$0 = y'' - xy' - x^2 y$

$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2}$

$n \leftrightarrow n+2$ $\qquad$ $n \leftrightarrow n-2$

$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n$

$= 2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} \right] x^n$

$2a_2 = 0$

$6a_3 - a_1 = 0$

$(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} = 0$; $n = 2, 3, 4, \ldots$

$a_{n+2} = \dfrac{n a_n + a_{n-2}}{(n+2)(n+1)}$; $n = 2, 3, 4, \ldots$

a_0 ARBITRARY.

a_1 ARBITRARY

$a_2 = 0$

$a_3 = \frac{1}{6} a_1$

The following problems make up the take-home portion of the test. These problems are due November 11, 2025. You must work on your own.

7. (9 points) Use the definition of the Laplace transform to find the transform of each function. Show all details.

(a) $f(t) = t, \quad t > 0$

$$F(s) = \int_0^\infty e^{-st} t \, dt = \frac{t}{s} e^{-st} \Big|_{t\to\infty}^{t=0} + \frac{1}{s} \underbrace{\int_0^\infty e^{-st} dt}_{\mathcal{L}\{1\}(s) = \frac{1}{s},\ s>0} = 0 - \lim_{t\to\infty} \frac{t}{s} e^{-st} + \frac{1}{s^2}$$

$u = t \qquad du = dt$

$dv = e^{-st} dt \qquad v = -\frac{1}{s} e^{-st}$

DID IT IN CLASS!

$$= \boxed{\frac{1}{s^2},\ s > 0}$$

$$\underbrace{\lim_{t\to\infty} \frac{t}{s e^{st}} = \lim_{t\to\infty}}_{\text{L'HOPITAL'S RULE}} \frac{1}{s^2 e^{st}} = 0,\ s > 0$$

(b) $f(t) = e^{-7t}, \quad t > 0$

$$F(s) = \int_0^\infty e^{-st} e^{-7t} dt = \int_0^\infty e^{-(s+7)t} dt = \frac{1}{s+7} e^{-(s+7)t} \Big|_{t\to\infty}^{t=0}$$

$$= \frac{1}{s+7} - \underbrace{\lim_{t\to\infty} e^{-(s+7)t}}_{\text{LIMIT IS 0 PROVIDED } s>-7}$$

$$= \boxed{\frac{1}{s+7},\ s > -7}$$

(c) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ 2, & 1 < t \le 3 \\ 0, & t \ge 3 \end{cases}$

$$F(s) = \int_0^1 e^{-st} dt + \int_1^3 2e^{-st} dt$$

$$= \frac{1}{s} e^{-st} \Big|_{t=1}^{t=0} + \frac{2}{s} e^{-st} \Big|_{t=3}^{t=1} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{2e^{-s}}{s} - \frac{2e^{-3s}}{s}$$

$$= \boxed{\frac{1 + e^{-s} - 2e^{-3s}}{s}}$$

HOMO EQUATION: $y'' + y = 0$

CHAR. EQUATION: $r^2 + 1 = 0$ $\rightarrow$ $r = \pm i \Rightarrow \alpha = 0,\ \beta = 1$

$y_c(t) = C_1 \cos t + C_2 \sin t$

8. (10 points) Consider the initial value problem: $y'' + y = \cos \gamma t;\quad y(0) = 0,\ y'(0) = 0.$

(a) Solve the IVP when $\gamma = 1$.

$g(t) = \cos t \Rightarrow y_p(t) = t\,(A\cos t + B\sin t)$

$y_p'(t) = A\cos t - At\sin t + B\sin t + Bt\cos t$

$y_p''(t) = -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t$

$y_p''(t) + y_p(t) = -2A\sin t + 2B\cos t$

$= \cos t \Rightarrow A = 0,\ B = \frac{1}{2} \Rightarrow y_p(t) = \frac{1}{2} t \sin t$

$y(t) = C_1\cos t + C_2 \sin t + \frac{1}{2} t \sin t$

$y(0) = 0 \Rightarrow C_1 = 0,\ y'(0) = 0 \Rightarrow C_2 = 0$

$\boxed{y_p(t) = \frac{1}{2} t \sin t}$

(b) Solve the IVP assuming $\gamma \neq 1$.

$g(t) = \cos \gamma t,\ \gamma \neq 1$

$y_p(t) = A\cos\gamma t + B\sin\gamma t$

$y_p'(t) = -A\gamma\sin\gamma t + B\gamma\cos\gamma t$

$y_p''(t) = -A\gamma^2\cos\gamma t - B\gamma^2\sin\gamma t$

$y_p''(t) + y_p(t) = A(1-\gamma^2)\cos\gamma t + B(1-\gamma^2)\sin\gamma t$

$= \cos\gamma t \Rightarrow B = 0,\ A = \frac{1}{1-\gamma^2} \Rightarrow y_p(t) = \frac{1}{1-\gamma^2}\cos\gamma t$

$y(t) = C_1\cos t + C_2\sin t + \frac{1}{1-\gamma^2}\cos\gamma t$

$y(0) = 0 \Rightarrow C_1 = -\frac{1}{1-\gamma^2},\ y'(0) = 0 \Rightarrow C_2 = 0$

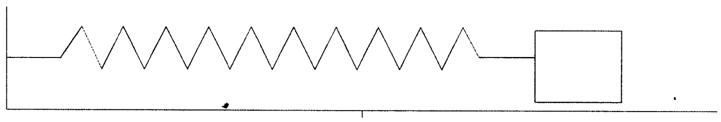
$\boxed{y_p(t) = \frac{1}{1-\gamma^2}\cos\gamma t - \frac{1}{1-\gamma^2}\cos t}$

(c) Use L'Hôpital's rule to show that as $\gamma \to 1$, your solution from part (b) approaches your solution from part (a).

$$\lim_{\gamma\to 1} \underbrace{\frac{1}{1-\gamma^2}(\cos\gamma t - \cos t)}_{\text{L'HOPITAL'S RULE}} \overset{0/0}{=} \lim_{\gamma\to 1} \frac{-t\sin\gamma t}{-2\gamma} = \frac{1}{2} t \sin t \quad \checkmark$$

DERIVATIVES W.R.T. γ.

9. (6 points) A 1-kg mass is attached to a spring with spring constant 4 N/m. The damping constant for the system is 0.1 N-sec/m. The mass is moved 1 m to the **left** of equilibrium (compressing the spring) and pushed to the **right** at 1 m/sec. At the moment the mass is pushed, the periodic external force $F(t) = \cos 2t$ is applied.



$x = 0$
(Equilibrium)

(a) Set up the initial value problem that describes the motion of the mass.

$$\boxed{x'' + 0.1x' + 4x = \cos 2t; \quad x(0) = -1, \ x'(0) = 1}$$

(b) Use SageMath (or some other CAS) to solve the initial value problem. (If you need help with the SageMath syntax, see the posted lecture notes for section 2.6.)

$$\boxed{x(t) = e^{-t/20}\left[-\cos\left(\frac{\sqrt{1599}}{20}t\right) - \frac{181}{\sqrt{1599}}\sin\left(\frac{\sqrt{1599}}{20}t\right)\right] + 5\sin(2t)}$$

(c) Once you have found your solution, identify the complementary solution (the transient part) and the particular solution (the steady-state part).

$$\boxed{x_c(t) = e^{-t/20}\left[-\cos\left(\frac{\sqrt{1599}}{20}t\right) - \frac{181}{\sqrt{1599}}\sin\left(\frac{\sqrt{1599}}{20}t\right)\right] = \text{TRANSIENT PART}}$$

$$\boxed{x_p(t) = 5\sin 2t = \text{STEADY STATE PART}}$$

(d) What is the angular frequency of the transient part? (Earlier in the semester, we called this a pseudo-frequency.) What is the angular frequency of the steady-state part?

$$\boxed{\beta = \frac{\sqrt{1599}}{20} \approx 1.999375 \text{ rad/s}} \qquad \boxed{\gamma = 2 \text{ rad/s}}$$

(e) Compute the gain factor for this system.

$F(t) = \cos 2t$ $\qquad$ $x_p(t) = 5\sin 2t$ $\qquad$ GAIN FACTOR $= 5$

Amp $= 1$ $\qquad$ Amp $= 5$

(f) Compute the resonance frequency for this system.

$$\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \boxed{\sqrt{3.995} \approx 1.99875}$$