Math 240 - Test 3

November 6, 2025

Name ______________________

Score __________

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

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1. (16 points) Find the general solution: $y'' + 4y' - 5y = (3+t)e^t$

2. (18 points) For $t > 0$, consider the equation

$$ty'' + (1 - 2t)y' + (t - 1)y = te^t.$$

The functions $y_1(t) = e^t$ and $y_2(t) = e^t \ln t$ are solutions of the corresponding homogeneous equation.

(a) Compute the Wronskian of y_1 and y_2 and confirm that these functions are linearly independent.

(b) Use variation of parameters to find a particular solution. (Hint: Be sure to divide by t before applying the variation of parameters formulas.)

(c) What is the general solution?

3. (3 points) Determine all singular points of the equation $xy'' + \frac{e^x}{x+5}y' + (\cos x)y = 0$.

4. (9 points) Find a minimum value for the radius of convergence of a power series solution centered at $x = c$.

   (a) $(x+1)y'' - 3xy' + 2y = 0, \qquad c = 1$

   (b) $y'' - xy' - 3y = 0, \qquad c = 2$

   (c) $(x^2 - 5x + 6)y'' - 3xy' - y = 0, \qquad c = 0$

5. (19 points) Consider the equation $(x^2+1)y'' - xy' + y = 0$. Notice that a power series solution centered at $x = 0$ has radius of convergence of at least 1.

   (a) Find the complete recurrence relation for the power series solution centered at $x = 0$.

   (b) One of the two linearly independent solutions is a very simple polynomial. Find it. Then find the first four nonzero terms of the second linearly independent solution.

   (c) Find the solution that satisfies $y(0) = 0$ and $y'(0) = 5$.

6. (10 points) Find the complete recurrence relation for the power series solution centered at $x = 0$.

$$y'' - xy' - x^2 y = 0$$

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The following problems make up the take-home portion of the test. These problems are due November 11, 2025. You must work on your own.

7. (9 points) Use the definition of the Laplace transform to find the transform of each function. Show all details.

(a) $f(t) = t, \quad t > 0$

(b) $f(t) = e^{-7t}, \quad t > 0$

(c) $f(t) = \begin{cases} 1, & 0 < t \leq 1 \\ 2, & 1 < t \leq 3 \\ 0, & t \geq 3 \end{cases}$

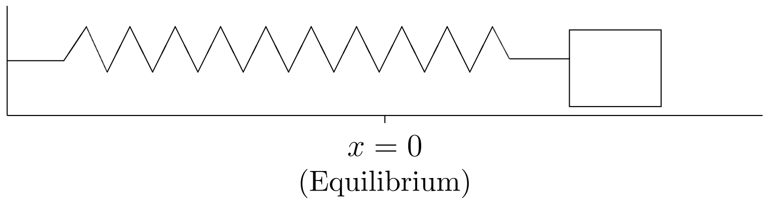
8. (10 points) Consider the initial value problem: $y'' + y = \cos \gamma t;\quad y(0) = 0,\ y'(0) = 0.$ The complementary solution is $y_c(t) = c_1 \cos t + c_2 \sin t$. You need not bother deriving it below.

(a) Solve the IVP when $\gamma = 1$.

(b) Solve the IVP assuming $\gamma \neq 1$.

(c) Use L'Hôpital's rule to show that as $\gamma \to 1$, your solution from part (b) approaches your solution from part (a).

9. (6 points) A 1-kg mass is attached to a spring with spring constant 4 N/m. The damping constant for the system is 0.1 N-sec/m. The mass is moved 1 m to the **left** of equilibrium (compressing the spring) and pushed to the **right** at 1 m/sec. At the moment the mass is pushed, the periodic external force $F(t) = \cos 2t$ is applied.



(a) Set up the initial value problem that describes the motion of the mass.

(b) Use SageMath (or some other CAS) to solve the initial value problem. (If you need help with the SageMath syntax, see the posted lecture notes for section 2.6.)

(c) Once you have found your solution, identify the complementary solution (the transient part) and the particular solution (the steady-state part).

(d) What is the angular frequency of the transient part? (Earlier in the semester, we called this a pseudo-frequency.) What is the angular frequency of the steady-state part?

(e) Compute the gain factor for this system.

(f) Compute the resonance frequency for this system.