Math 240 - Final Exam A

December 5, 2025

Name Key

Score ________

Show all work to receive full credit. You must work individually. This test is due December 11. All integration must be done by hand (showing work).

---

1. (10 points) A large tank initially contains 1000 L of a salt solution in which 0.5 kg of salt are dissolved. A salt solution containing 0.02 kg of salt per liter enters the tank at 4 L/min and is uniformly mixed. The mixed solution leaves the tank at 2 L/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. When will the concentration of salt in the tank reach 0.01 kg/L?

$0.02\ \text{kg/L} \cdot 4\ \text{L/min} = 0.08\ \text{kg/min}$

$V(0) = 1000$

$A(0) = 0.5$

$A(t)$ = AMOUNT OF SALT (kg) AT TIME t

$V(t) = 1000 + 2t$ = VOLUME (L) AT TIME t

$$2\ \text{L/min} \cdot \frac{A(t)}{V(t)}\ \text{kg/L} = \frac{2A}{1000+2t}\ \text{kg/min}$$

$$\frac{dA}{dt} = 0.08 - \frac{A}{500+t}, \quad A(0) = 0.5$$

$$\frac{dA}{dt} + \frac{1}{500+t} A = 0.08$$

$$\mu(t) = e^{\int \frac{1}{500+t}\,dt} = 500+t, \quad t \geq 0$$

$$(500+t)A = \int 0.08(500+t)\,dt = \frac{0.08}{2}(500+t)^2 + C$$

$$A(t) = 0.04(500+t) + \frac{C}{500+t}$$

$$A(0) = 0.5 \Rightarrow 20 + \frac{C}{500} = 0.5 \Rightarrow C = -9750$$

$$\boxed{A(t) = 0.04(500+t) - \frac{9750}{500+t} = \frac{0.04t^2 + 40t + 250}{500+t}}$$

$$\frac{A(t)}{V(t)} = \frac{0.04t^2 + 40t + 250}{2(500+t)^2} = 0.01$$

$$\Rightarrow 0.04t^2 + 40t + 250 = 0.02(500+t)^2$$

OVER $\longrightarrow$

$$0.04t^2 + 40t + 250 = 5000 + 20t + 0.02t^2$$

$$0.02t^2 + 20t - 4750 = 0$$

$$t = \frac{-20 \pm \sqrt{400 + 380}}{0.04} \approx 198.2 \text{ min} \quad \text{or} \quad \cancel{-1198.2 \text{ min}}$$

$$\boxed{t \approx 198.2 \text{ min}}$$

2. (10 points) Use Laplace transform methods to solve the following equation.

$$ty'' + (2t - 3)y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 0$$

LET $Y(s) = \mathcal{L}\{y(t)\}(s)$.

$$-\frac{d}{ds}\left(s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right) - 2\frac{d}{ds}\left(sY(s) - \cancel{y(0)}^{0}\right) - 3\left(sY(s) - \cancel{y(0)}^{0}\right) + 2Y(s) = 0$$

$$-2sY - s^2Y' - 2Y - 2sY' - 3sY + 2Y = 0$$

$$(-s^2 - 2s)Y' - 5sY = 0$$

$$Y' + \frac{5}{s+2}Y = 0. \Rightarrow \frac{dY}{ds} = -\frac{5}{s+2}Y \Rightarrow \frac{dY}{Y} = -\frac{5}{s+2}ds.$$

$$\Rightarrow \ln|Y| = -5\ln|s+2|.$$

$$\Rightarrow Y(s) = \frac{C}{(s+2)^5}$$

NOT SAME C !

$$\boxed{y(t) = Ct^4 e^{-2t}}$$

3. (10 points) Solve the following one-dimensional heat equation with Dirichlet boundary conditions. Do not derive the solution method–just use the result we derived in class. (See Theorem 1 on page 593.)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 10, \quad t \geq 0,$$

$$u(0,t) = u(10,t) = 0,$$

$$u(x,0) = 100 - 10x, \quad 0 \leq x \leq 10$$

$L = 10, \; k = \frac{1}{2}$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{\frac{-n^2\pi^2 t}{200}} \sin\left(\frac{n\pi}{10}x\right)$$

WHERE $\quad C_n = \frac{2}{10}\int_0^{10} (100-10x)\sin\left(\frac{n\pi}{10}x\right)dx$

SIGNS	U AND DERIVS	dv/dx AND ANTIS
+	$100-10x$	$\sin\left(\frac{n\pi}{10}x\right)$
−	-10	$-\frac{10}{n\pi}\cos\left(\frac{n\pi}{10}x\right)$
+	0	$-\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi}{10}x\right)$

$$= \frac{1}{5}\left[\frac{10(10x-100)}{n\pi}\cos\left(\frac{n\pi}{10}x\right) - \frac{1000}{n^2\pi^2}\sin\left(\frac{n\pi}{10}x\right)\right]_{x=0}^{x=10}$$

$$= \frac{1}{5}\left[0 - 0 + \frac{1000}{n\pi} + 0\right]$$

$$= \frac{200}{n\pi}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} e^{-\frac{n^2\pi^2 t}{200}} \sin\left(\frac{n\pi}{10}x\right)}$$