Math 240 - Final Exam B

December 11, 2025

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

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1. (10 points) According to Newton's law of cooling, the temperature T at time t of an object cooling in a medium of constant temperature M is described by the differential equation
$$\frac{dT}{dt} = k(M-T),$$
where k is some constant.

   (a) Solve the differential equation.

$$\frac{dT}{m-T} = k\,dt$$

$$-\ln|m-T| = kt + C_1$$

$$\ln|m-T| = -kt + C_2$$

$$|m-T| = C_3 e^{-kt}$$

$$m - T = C_4 e^{-kt}$$

$$\boxed{T(t) = m - Ce^{-kt}}$$

   (b) An object at 120°F is moved into a large room with an ambient temperature of 72°F. The object cools to 100°F in 6 min. Use your result from part (a) to find a formula for the temperature of the object at time t.

$T(0) = 120$, $m = 72$

$T(6) = 100$

$$T(t) = 72 - Ce^{-k(0)} = 120$$

$$\Rightarrow C = -48$$

$$\boxed{T(t) = 72 + 48e^{-kt}}$$

$$T(6) = 100 \Rightarrow 28 = 48e^{-6k}$$

$$\frac{28}{48} = e^{-6k} \Rightarrow \boxed{k = \frac{\ln\left(\frac{28}{48}\right)}{-6}}$$

   (c) When will the object reach 76°F?

$$76 = 72 + 48e^{-kt} \Rightarrow 4 = 48e^{-kt} \Rightarrow \frac{4}{48} = e^{-kt}$$

$$t = \frac{\ln\left(\frac{4}{48}\right)}{-k} \approx \boxed{27.66 \text{ min}}$$

2. (10 points) The vector force $M(x,y)\,\hat{\imath}+N(x,y)\,\hat{\jmath}$ is a *conservative* force if the differential equation $M(x,y)\,dx+N(x,y)\,dy=0$ is an exact differential equation.

Consider the equation $(y\cos x+y^2)\,dx+(\sin x+2xy-2y)\,dy=0$.

(a) Show that the equation is exact.

$M(x,y) = y\cos x + y^2$, $N(x,y) = \sin x + 2xy - 2y$

$$\frac{\partial M}{\partial y} = \cos x + 2y \qquad \frac{\partial N}{\partial x} = \cos x + 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$ EQUATION IS EXACT.

(b) When a force is conservative, we can find a *potential function* by solving the corresponding exact differential equation. Solve the exact equation above.

$$F_x(x,y) = y\cos x + y^2 \Rightarrow F(x,y) = y\sin x + xy^2 + g(y)$$

$$F_y(x,y) = \sin x + 2xy - 2y \Rightarrow F(x,y) = y\sin x + xy^2 - y^2 + h(x)$$

$$F(x,y) = y\sin x + xy^2 - y^2$$

THIS IS A POTENTIAL FUNCTION.

SOLUTION IS

$$y\sin x + xy^2 - y^2 = C$$

3. (10 points) Solve the initial value problem: $y' + y\cot x = 2x\csc x, \quad y(\pi/2) = \pi^2$

$$\mu(x) = e^{\int \cot x \, dx} = e^{\ln|\sin x|} = |\sin x|$$

BASED ON I.C.

LET'S ASSUME $\sin x > 0$

$$= \sin x$$

$$(\sin x)\, y = \int 2x \, dx$$

$$= x^2 + C$$

$$y(x) = \frac{x^2}{\sin x} + \frac{C}{\sin x}$$

$$y\left(\frac{\pi}{2}\right) = \pi^2 \Rightarrow \frac{\pi^2}{4} + C = \pi^2$$

$$\Rightarrow C = \frac{3\pi^2}{4}$$

$$\boxed{y(x) = \frac{x^2}{\sin x} + \frac{3\pi^2}{4\sin x}, \quad \sin x > 0}$$

4. (10 points) Find the general solution of each equation.

(a) $y^{(5)} - 8y^{(3)} + 16y' = 0$

CHAR. EQN: $r^5 - 8r^3 + 16r = r(r^4 - 8r^2 + 16)$

$r(r^2-4)^2 = r(r+2)^2(r-2)^2 = 0$

$r=0, r=-2, r=-2, r=2, r=2$

$y(x) = c_1 e^{0x} + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{2x} + c_5 x e^{2x}$

$$\boxed{y(x) = c_1 + (c_2 + c_3 x)e^{-2x} + (c_4 + c_5 x)e^{2x}}$$

(b) $x^2y'' + 9xy' + 17y = 0, \quad x > 0$

CAUCHY-EULER EQUATION. $x = e^t$ TRANSFORMS TO

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 17y = 0$$

CHAR EQN: $r^2 + 8r + 17 = 0$

$r^2 + 8r + 16 = -1$

$(r+4)^2 = -1 \Rightarrow r = -4 \pm i \Rightarrow \alpha = -4, \beta = 1$

$y(t) = c_1 e^{-4t}\cos t + c_2 e^{-4t}\sin t$

$t = \ln x$

$$\boxed{y(x) = \frac{c_1\cos(\ln x)}{x^4} + \frac{c_2\sin(\ln x)}{x^4}}$$

5. (10 points) Use undetermined coefficients to solve.

$$y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x; \qquad y(0) = 4, \quad y'(0) = -2$$

HOMO EQN: $y'' - 3y' + 2y = 0$

CHAR. EQN: $r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r = 2, r = 1$

$y_c(x) = c_1 e^{2x} + c_2 e^{x}$

NONHOMO EQN HAS $g(x) = 14\sin 2x - 18\cos 2x$

$y_p(x) = x^s (A\sin 2x + B\cos 2x)$ CHOOSE $s = 0$.

$y_p(x) = A\sin 2x + B\cos 2x$

$y_p'(x) = -2B\sin 2x + 2A\cos 2x$

$y_p''(x) = -4A\sin 2x - 4B\cos 2x$

$y_p'' - 3y_p' + 2y_p = 14\sin 2x - 18\cos 2x$

$\Downarrow$

$-4A + 6B + 2A = 14$

$-4B - 6A + 2B = -18$

$-A + 3B = 7$

$-3A - B = -9$

$-10A = -20 \Rightarrow A = 2$

$B = 3$

$y_p(x) = 2\sin 2x + 3\cos 2x$

$y(x) = c_1 e^{2x} + c_2 e^{x} + 2\sin 2x + 3\cos 2x$

$y(0) = 4 \Rightarrow c_1 + c_2 = 1$

$y'(0) = -2 \Rightarrow 2c_1 + c_2 = -6$

$-c_1 = 7$

$c_1 = -7 \qquad c_2 = 8$

$y(x) = 8e^{x} - 7e^{2x} + 2\sin 2x + 3\cos 2x$

6. (10 points) The equation

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$(1-x^2)y'' - xy' + 4y = 0$$

is an example of Chebyshev's equation of the first kind.

(a) Find the complete recurrence relation for a power series solution centered at $x = 0$.

$$0=\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{\text{REPLACE } n \text{ w/ } n+2} - \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^n}_{\text{CAN START AT } n=0.} - \underbrace{\sum_{n=1}^{\infty} n a_n x^n}_{\text{CAN START AT } n=0} + \sum_{n=0}^{\infty} 4a_n x^n$$

$$=\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n$$

$$=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n^2-4)a_n\right]x^n$$

$$\Rightarrow \quad (n+2)(n+1)a_{n+2} - (n^2-4)a_n = 0; \quad n=0,1,2,\ldots$$

$$a_{n+2} = \frac{(n^2-4)}{(n+2)(n+1)}a_n = \frac{n-2}{n+1}a_n; \quad n=0,1,2,\ldots$$

$$\boxed{a_{n+2} = \frac{n-2}{n+1}a_n; \quad n=0,1,2,3,\ldots \qquad a_0, a_1 \text{ ARB.}}$$

(b) One of the two linearly independent solutions is a polynomial of degree 2. Find it.

$a_0 = \text{ARB}$

$a_2 = -2a_0$

$a_4 = 0$

$a_6 = a_8 = \cdots = 0$

$$\boxed{y_1(x) = a_0(1-2x^2)}$$

(c) Find the first four nonzero terms of the second solution.

$a_1 = \text{ARB}$

$a_3 = -\frac{1}{2}a_1$

$a_5 = \frac{1}{4}a_3 = -\frac{1}{8}a_1$

$a_7 = \frac{1}{2}a_5 = -\frac{1}{16}a_1$

$$\boxed{y_2(x) = a_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \frac{1}{16}x^7 - \cdots\right)}$$

7. (10 points) Use Laplace transforms to solve the initial value problem.

$$y'' + 2y' - 15y = 6e^{-2t}; \qquad y(0) = 1, \quad y'(0) = 2$$

Let. $Y(s) = \mathcal{L}\{y(t)\}(s)$

$$s^2 Y(s) - s - 2 + 2sY(s) - 2 - 15Y(s) = \frac{6}{s+2}$$

$$\underbrace{(s^2 + 2s - 15)}_{(s+5)(s-3)} Y(s) = \frac{6}{s+2} + s + 4$$

$$Y(s) = \frac{6}{(s+5)(s-3)(s+2)} + \frac{s+4}{(s+5)(s-3)}$$

$$= \frac{A}{s+5} + \frac{B}{s-3} + \frac{C}{s+2} + \frac{D}{s+5} + \frac{E}{s-3}$$

Cover up gives $A = \frac{6}{24}$, $B = \frac{6}{40}$, $C = -\frac{6}{15}$, $D = \frac{1}{8}$, $E = \frac{7}{8}$

$$Y(s) = \frac{1/4}{s+5} + \frac{3/20}{s-3} - \frac{2/5}{s+2} + \frac{1/8}{s+5} + \frac{7/8}{s-3}$$

$$= \frac{3/8}{s+5} + \frac{41/40}{s-3} - \frac{2/5}{s+2}$$

$$\boxed{y(t) = \frac{3}{8}e^{-5t} + \frac{41}{40}e^{3t} - \frac{2}{5}e^{-2t}}$$