Math 240 - Assignment 1

August 20, 2026

Name KEY ______________

Score ______________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due August 27.

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1. According to Torricelli's law, the height, h, of water draining under the force of gravity from a cylindrical tank is described by the equation $\frac{dh}{dt} = -k\sqrt{h}$, where k is a constant that depends on the cross-sectional area of the tank. Say whether the equation is linear or nonlinear, state its order, and say which variables are independent and dependent.

   Solution

   The ordinary differential equation is a 1st-order equation. The independent variable is t, and the dependent variable is h. The equation is nonlinear because of the square root of the dependent variable.

2. The differential equation $x'' + \omega^2 x = 0$ arises in applications involving undamped oscillations.

   (a) State the order of the equation and say which variable is independent and which is dependent.

   Solution

   The ordinary differential equation is a 2nd-order equation. The independent variable is not named, but let's say it is t (see part (c)). The dependent variable is x.

   (b) Say whether the equation is linear or nonlinear and explain how you know.

   Solution

   The equation is linear. It matches the form of a 2nd-order linear eqution:

   $$a(t)x'' + b(t)x' + c(t)x = f(t),$$

   where $a(t) = 1$, $b(t) = 0$, $c(t) = \omega^2$ (constant function), and $f(t) = 0$.

   (c) Verify by substitution that $x(t) = A\cos\omega t + B\sin\omega t$ is a solution for any arbitrary constants A and B.

   Solution

   $$x(t) = A\cos\omega t + B\sin\omega t$$
   $$x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$
   $$x''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

   It follows immediately that $x'' + \omega^2 x = 0$.

(d) Use the solution from part (c) to find the particular solution that satisfies the initial conditions $x(0) = 2$ and $x'(0) = -1$.

Solution

Refer to $x(t)$ and $x'(t)$ above.

$$x(0) = A \text{ and } x(0) = 2 \quad \Longrightarrow \quad A = 2$$

$$x'(0) = B\omega \text{ and } x'(0) = -1 \quad \Longrightarrow \quad B = -1/\omega$$

The solution is $x(t) = 2\cos\omega t - \frac{1}{\omega}\sin\omega t$.

3. Consider the differential equation $2xy\dfrac{dy}{dx} = 4x^2 + 3y^2$.

(a) State the order of the equation and say which variable is independent and which is dependent.

Solution

The ordinary differential equation is a 1st-order equation. The independent variable is x, and the dependent variable is y.

(b) Say whether the equation is linear or nonlinear and explain how you know.

Solution

The equation is nonlinear because of both the product $y\dfrac{dy}{dx}$ and the power y^2.

(c) Without attempting to solve the equation, verify that $y^2 + 4x^2 = kx^3$ is a solution for any arbitrary constant k.

Solution

First notice that $y^2 = kx^3 - 4x^2$, so that

$$\frac{d}{dx}y^2 = \frac{d}{dx}(kx^3 - 4x^2)$$

or

$$2y\frac{dy}{dx} = 3kx^2 - 8x,$$

from which it follows that

$$2xy\frac{dy}{dx} = 3kx^3 - 8x^2.$$

Now notice that the right-hand side of the equation above is the same as

$$4x^2 + 3(kx^3 - 4x^2).$$

In other words, $2xy\dfrac{dy}{dx} = 4x^2 + 3y^2$.

(d) Is the solution given in part (c) an explicit solution or an implicit solution?

Solution

Since the dependent variable y is not explicitly solved for, the solution is implicit.

(e) Use the solution from part (c) to find the particular solution that satisfies $y(1) = 5$.

Solution

Since $y(1) = 5$, we must have

$$(5)^2 + 4(1)^2 = k(1)^3,$$

which means $k = 29$.

4. A plant sprouts at time $t = 0$ and grows in such a way that the instantaneous rate of change of its height, h, is proportional to the average rate of change of its height over its lifetime. Write a differential equation that models the height of the plant.

Solution

$$\frac{dh}{dt} = k\,\frac{h(t) - h(0)}{t - 0} \qquad \text{or} \qquad \frac{dh}{dt} = \frac{kh}{t}.$$

5. The equation below is the *Black-Scholes-Merton equation*, which describes the price evolution of financial derivatives. (Black and Scholes received the 1997 Nobel Prize in Economics.) Classify the differential equation by saying whether it is ordinary or partial, linear or nonlinear. Also give its order and name the dependent and independent variables.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Solution

This equation is a 2nd order, linear, partial differential equation. The dependent variable is V, and the independent variables are t and S. The symbols r and σ stand for constants.

6. Find the general solution: $\quad x\,\dfrac{dy}{dx} = \dfrac{1}{x-1}$

Solution

$$\frac{dy}{dx} = \frac{1}{x(x-1)}$$

Partial fraction decomposition...

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \qquad \Longrightarrow \qquad 1 = A(x-1) + B(x)$$

It follows that $A + B = 0$ and $-A = 1$, from which we get $A = -1$ and $B = 1$. Therefore

$$\frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}.$$

Upon integrating, we have

$$y(x) = -\ln|x| + \ln|x-1| + C \quad \text{or} \quad y(x) = \ln\left|\frac{x-1}{x}\right| + C.$$

7. For which values of r is $y(x) = e^{rx}$ a solution of $y'' - 2y' - 15y = 0$?

   Solution

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

   Substituting into the equation gives

$$y'' - 2y' - 15y = (r^2 - 2r - 15)e^{rx} = 0.$$

   This equation will be true if and only if $r^2 - 2r - 15 = (r-5)(r+3) = 0$.

   It follows that $r = 5$ or $r = -3$.

8. The graph of $y = f(x)$ has the following property:

   The normal line through the point (x, y) passes through the point $(1, 2)$.

   Write a differential equation that describes the family of such curves.

   Solution

$$\text{Slope of normal line through } (x, y) = -\frac{1}{dy/dx} = \frac{y-2}{x-1}$$

$$\frac{dy}{dx} = -\frac{x-1}{y-2}$$

9. Solve the initial value problem: $\dfrac{dy}{dx} = 3x\sqrt{x^2+4}, \quad y(0) = 4.$

   Solution

$$y(x) = \int 3x\sqrt{x^2+4}\,dx$$

$$u = x^2 + 4, \qquad du = 2x\,dx, \qquad \frac{3}{2}du = 3x\,dx$$

$$\int 3x\sqrt{x^2+4}\,dx = \frac{3}{2}\int u^{1/2}\,du = u^{3/2}+C = (x^2+4)^{3/2}+C$$

$$y(0)=4 \quad\Longrightarrow\quad 4^{3/2}+C=4 \quad\Longrightarrow\quad C=-4$$

$$y(x) = (x^2+4)^{3/2}-4$$

10. Solve the initial value problem: $\dfrac{1}{x\sin(2x)}\dfrac{dy}{dx} = 1, \quad y(0)=4.$

<u>Solution</u>

$$\frac{dy}{dx} = x\sin(2x)$$

$$y(x) = \int x\sin(2x)\,dx$$

$$u=x, \quad dv=\sin(2x), \quad du=dx, \quad v=-\frac{1}{2}\cos(2x)$$

$$y(x) = -\frac{x}{2}\cos(2x)+\frac{1}{2}\int\cos(2x)\,dx$$

$$y(x) = -\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$$

$$y(0)=4 \quad\Longrightarrow\quad C=4$$

$$y(x) = -\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+4$$