MTH 240 Assignment 2 Key

1) $\frac{dy}{dx} = y^{1/3}, \; y(0) = 0, \quad x \geq 0$

a) $y^{-1/3}\,dy = dx$

$\frac{3}{2}y^{2/3} = x + C_1$

$y(x) = \sqrt{\left(\frac{2}{3}x + C_2\right)^3} \qquad y(0) = 0 \Rightarrow C_2 = 0$

$$\boxed{y(x) = \sqrt{\frac{8x^3}{27}}}$$

b) $\boxed{y(x) \equiv 0}$ IS A SINGULAR SOLUTION, ALSO SATISFYING $y(0) = 0$.

2) $f(x,y) = y^{1/3}$ IS CONTINUOUS EVERYWHERE.

WE EXPECT A SOLUTION THROUGH <u>ANY</u> INITIAL POINT.

$f_y(x,y) = \frac{1}{3}y^{-2/3}$ IS NOT DEFINED WHEN $y = 0$ AND IS THEREFORE NOT CONTINUOUS IN ANY NEIGHBORHOOD OF $(0,0)$.

WE CANNOT GUARANTEE A UNIQUE SOLUTION THROUGH $(0,0)$.

3.) $\frac{dy}{dx} = \ln(1+y^2)$, $y(0) = 0$

$f(x,y) = \ln(1+y^2)$ IS CONTINUOUS EVERYWHERE.

$f_y(x,y) = \frac{2y}{1+y^2}$ IS CONTINUOUS EVERYWHERE.

THERE IS A UNIQUE SOLUTION THROUGH <u>ANY</u> INITIAL POINT. (B)

4.) $(y')^2 - xy' + y = 0$, $y(2) = 1$

By QUAD. FORMULA, $y' = \frac{x \pm \sqrt{x^2 - 4y}}{2}$.

$f(x,y) = \frac{x \pm \sqrt{x^2-4y}}{2}$

FOR EITHER CHOICE OF $\pm$, f IS NOT DEFINED WHEN $x^2 - 4y < 0$. THIS WILL HAPPEN IN ANY NEIGHBORHOOD OF $(2,1)$.

$f(x,y)$ IS NOT CONTINUOUS ON A RECTANGLE AROUND $(2,1)$. A SOLUTION IS NOT GUARANTEED. (C)

5.) $\frac{dy}{dx} = (x^2+1)(y-2)^{4/5}$, $y(1) = -2$

$f(x,y) = (x^2+1)(y-2)^{4/5}$ IS CONTINUOUS EVERYWHERE.

$f_y(x,y) = \frac{4}{5}(x^2+1)(y-2)^{-1/5}$ IS CONTINUOUS EVERYWHERE THAT $y \neq 2$.

AROUND OUR INITIAL POINT $(1,-2)$, BOTH f AND f_y ARE CONTINUOUS. THERE IS A UNIQUE SOLUTION THROUGH $(1,-2)$. (B)

6.) $\frac{dy}{dx} = x^2 - y$. SEE SLOPE FIELD ON NEXT PAGE.

a) $\left.\frac{dy}{dx}\right|_{(1,1)} = (1)^2 - 1 = \boxed{0}$

b) THE SOLUTION CURVE THROUGH $(-3,2)$ WILL HAVE BOTH A REL. MAX AND A REL MIN. DRAW THE ROUGH SKETCH AND YOU SHOULD HAVE A MAX NEAR $X=-2$ AND A MIN NEAR $X=1$.

c) THE SOLUTION CURVE THROUGH $(0,-4)$ WILL NOT HAVE MAX/MIN. IT LOOKS LIKE THAT SOLUTION WILL ALWAYS BE INCREASING (QUICKLY).

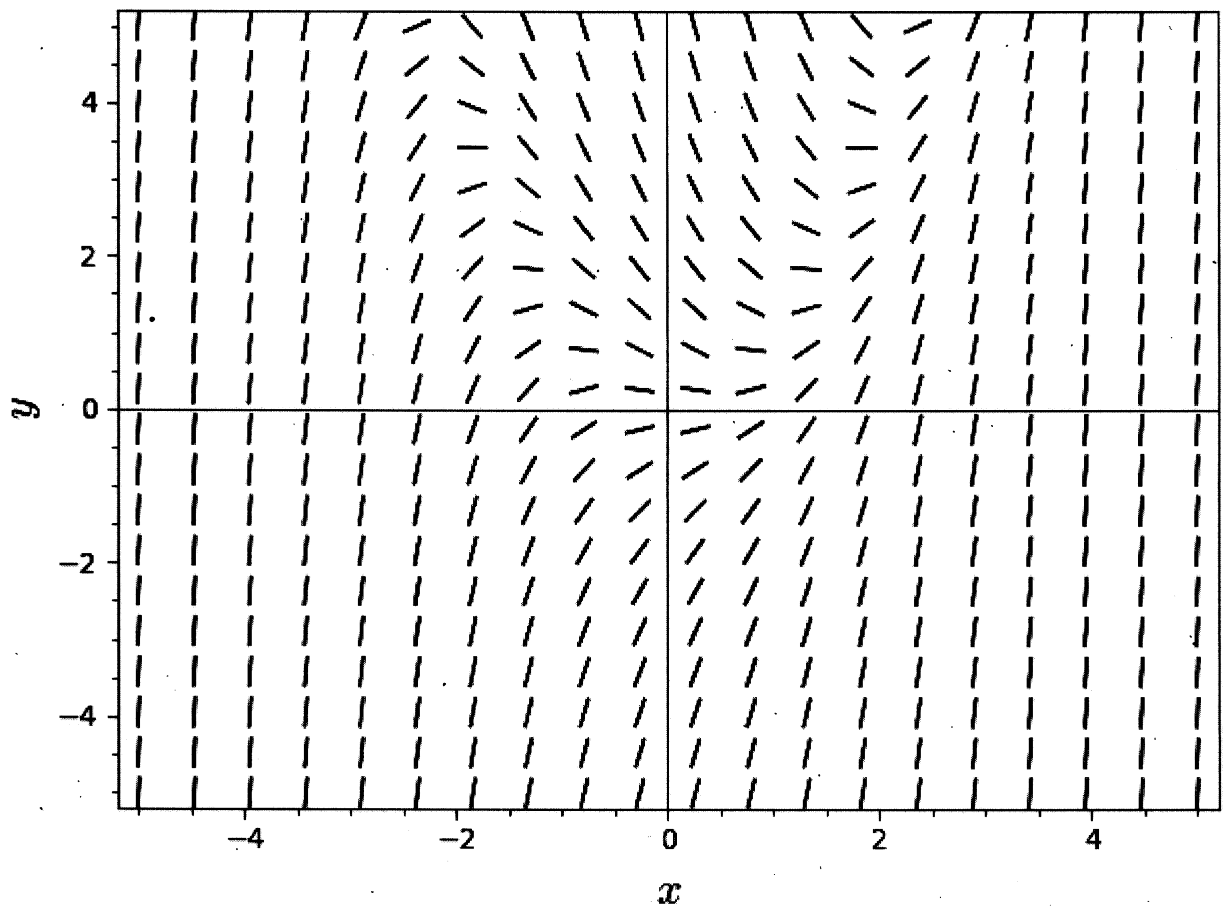
d) IT LOOKS LIKE $\lim_{x\to\infty} y(x) = \infty$ FOR ANY SOLUTION $y(x)$.

e) ① FOR $\frac{dy}{dx} = \frac{-y}{x}$, ALL SLOPES WOULD BE NEG. IN QUAD I.

② FOR $\frac{dy}{dx} = \frac{-y}{x}$, SOLUTION CURVES WOULD HAVE VERTICAL TANGENTS ALONG $X=0$.

} NOT HAPPENING!

4



7.) $\frac{dy}{dx} = y - x - 1, \quad y(0) = 1$

WITH $h = 0.1$ ...

$$y_1 = y_0 + 0.1\,(y_0 - x_0 - 1)$$

$$= 1 + 0.1\,(1 - 0 - 1) = 1$$

$$x_1 = 0.1$$

$$y_2 = y_1 + 0.1\,(y_1 - x_1 - 1)$$

$$= 1 + 0.1\,(1 - 0.1 - 1) = 0.99$$

$$x_2 = 0.2$$

$$y_3 = y_2 + 0.1\,(y_2 - x_2 - 1)$$

$$= 0.99 + 0.1\,(0.99 - 0.2 - 1)$$

$$= 0.969$$

$$\boxed{y(0.3) \approx 0.969}$$

WITH $h = 0.01$, SEE ATTACHED SHEETS.

$$\boxed{y(0.3) \approx 0.95215}$$

SageMathCell

Type your own Sage computation below and then click "Evaluate".

```
for i in range( steps ):
    yy = yy + h * f(xx,yy)
    xx = xx + h
    xy.append([xx,yy])

print("\nEnding values:")
print("x = ", xx, ", y = ", yy)

print("\n", *xy, sep="\n" )
plt.scatter(*zip(*xy))
plt.show()
```

Evaluate

Share

```
Starting values:
x  =  0.0 , y =  1.0

Ending values:
x =  0.3000000000000001 , y =  0.9521510846670944


[0.0, 1.0]
[0.01, 1.0]
[0.02, 0.9999]
[0.03, 0.999699]
[0.04, 0.99939599]
[0.05, 0.9989899499]
[0.060000000000000005, 0.998479849399]
[0.07, 0.99786464789299]
[0.08, 0.99714329437192]
[0.09, 0.9963147273156392]
[0.09999999999999999, 0.9953778745887956]
[0.10999999999999999, 0.9943316533346835]
[0.11999999999999998, 0.9931749698680303]
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[0.3000000000000001, 0.9521510846670944]
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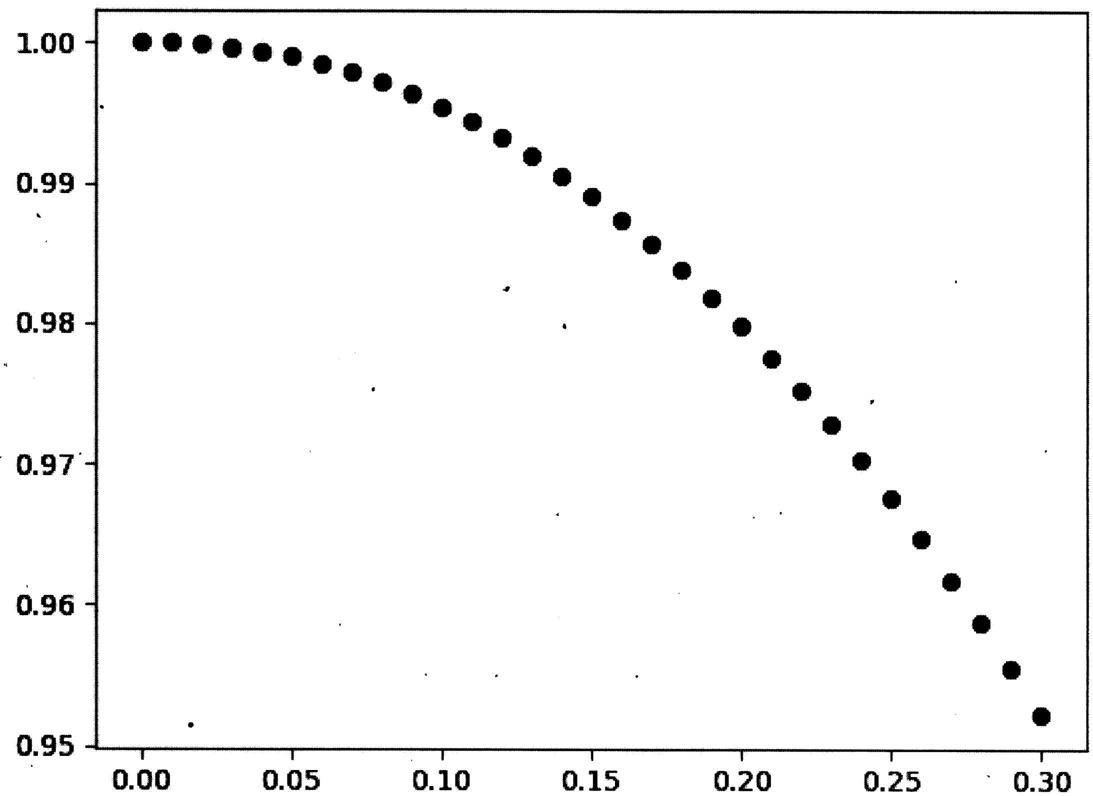

Type your own Sage computation below and then click "Evaluate".

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1
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Evaluate

Type your own Sage computation below and then click "Evaluate".

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1
```

Evaluate

Need some help?

Click here for the SageMath website. Click here for some quick reference material.

8.) $\frac{dy}{dx} = x + \ln(y), \quad y(1) = 2$

WITH $h = 0.25 \ldots$

$y_1 = 2 + (0.25)(1 + \ln 2)$

≈ 2.423287

$x_1 = 1.25$

$y_2 = y_1 + (0.25)(1.25 + \ln(y_1))$

≈ 2.957068

$x_2 = 1.5$

$y_3 = y_2 + (0.25)(1.5 + \ln(y_2))$

≈ 3.603118

$x_3 = 1.75$

$y_4 = y_3 + (0.25)(1.75 + \ln(y_3))$

≈ 4.361069

$\boxed{y(2) \approx 4.361}$

9.) $\frac{dv}{dt} = -kv^2$

$$\frac{1}{v^2}\, dv = -k\, dt$$

$$-\frac{1}{v} = -kt + C_1$$

$$\frac{1}{v} = kt + C_2$$

$$v = \frac{1}{kt + C_2} \qquad \boxed{v(t) = \frac{1}{kt + C}}$$

10.) $2y \frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \quad y(5) = 2$

$$2y\, dy = \frac{x}{\sqrt{x^2-16}}\, dx$$

$$y^2 = \int \frac{x}{\sqrt{x^2-16}}\, dx = \frac{1}{2}\int u^{-1/2}\, du = u^{1/2} + C$$

$$= \sqrt{x^2-16} + C$$

$u = x^2 - 16$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$.

$$y(x) = \sqrt{C + \sqrt{x^2-16}}$$

TAKING POS SQUARE ROOT BASED ON IC.

$$y(5) = 2 \Rightarrow \sqrt{C+3} = 2$$

$$\Rightarrow C = 1$$

$$\boxed{y(x) = \sqrt{1 + \sqrt{x^2-16}}}$$

11.) $\sqrt{y}\,dx + (1+x)\,dy = 0,\ y(0)=1$

$(1+x)\,dy = -\sqrt{y}\,dx$

$y^{-1/2}\,dy = \dfrac{-1}{1+x}\,dx$

$\int y^{-1/2}\,dy = \int -\dfrac{1}{1+x}\,dx$

$2\sqrt{y} = -\ln|1+x| + C_1$

$\sqrt{y} = -\frac{1}{2}\ln|1+x| + C_2$

$y(x) = \left(C - \frac{1}{2}\ln|1+x|\right)^2$

$y(0)=1 \Rightarrow C=1$

$\boxed{y(x) = \left(1 - \frac{1}{2}\ln|1+x|\right)^2}$

OR

$y(x) = \left(1 - \ln\sqrt{|1+x|}\right)^2$

12.) $v(t) = \dfrac{1}{kt+C}$ (SEE PROB 9)

$v(0) = 40 \Rightarrow \dfrac{1}{C} = 40 \Rightarrow C = \dfrac{1}{40}$

$v(t) = \dfrac{40}{40kt+1}$

$v(10) = 20 \Rightarrow 20 = \dfrac{40}{400k+1} \Rightarrow 400k+1 = 2$

$k = \dfrac{1}{400}$

$v(t) = \dfrac{40}{\frac{1}{10}t+1} = \dfrac{400}{t+10}$

$x(t) = \int \dfrac{400}{t+10}\,dt = 400\ln(t+10) + C$

$x(60) - x(0) = 400\ln 70 - 400\ln 10 \approx \boxed{778.4 \text{ ft}}$