

MTH 240 Assignment #4 Key

①

1.) $x \frac{dy}{dx} - y = 2x^2y, y(1) = 1$

a) SEPARABLE: $x dy = (2x^2y + y) dx$

$$\frac{1}{y} dy = \frac{2x^2 + 1}{x} dx$$

$$\int \frac{1}{y} dy = \int (2x + \frac{1}{x}) dx$$

$$\ln |y| = x^2 + \ln |x| + C_1$$

$$|y| = e^{x^2 + \ln |x| + C_1} = C_2 |x| e^{x^2}$$

$$y(x) = C_3 x e^{x^2}, y(1) = 1 \Rightarrow 1 = C_3 e \Rightarrow C_3 = \frac{1}{e}$$

$$y(x) = x e^{x^2 - 1}$$

b) LINEAR: $\frac{dy}{dx} - \left(\frac{2x^2 + 1}{x}\right)y = 0$

$$\mu(x) = e^{-\int (2x + \frac{1}{x}) dx} = e^{-x^2 - \ln |x|} = \frac{e^{-x^2}}{|x|} = \frac{e^{-x^2}}{x}, x > 0$$

$$\frac{e^{-x^2}}{x} y = \int 0 dx = C$$

$$y(x) = C x e^{x^2}, y(1) = 1 \Rightarrow 1 = C e \Rightarrow C = \frac{1}{e}$$

$$y(x) = x e^{x^2 - 1}$$

Problem #1 CONTINUED...

c) Bernoulli: $\frac{dy}{dx} - \frac{1}{x}y = 2xy$

$$y^{-1} \frac{dy}{dx} - \frac{1}{x} = 2x$$

Assume $y > 0$ and let $u = \ln y$

$$\frac{du}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\begin{aligned} \frac{du}{dx} &= 2x + \frac{1}{x} \Rightarrow u(x) = x^2 + \ln|x| + C_1 \\ &= x^2 + \ln x + C_1, \quad x > 0 \end{aligned}$$

$$y = e^u$$

$$y = e^{x^2 + \ln x + C_1}$$

$$y(x) = C_2 x e^{x^2}$$

$$y(1) = 1 \Rightarrow 1 = C_2 e \Rightarrow C_2 = \frac{1}{e}$$

$$y(x) = x e^{x^2 - 1}$$

2.) $\frac{dy}{dx} = \frac{y+x}{x-y}, \quad y(1) = 0$

$\frac{dy}{dx} = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}} \quad \text{Let } u = \frac{y}{x} \text{ so that } ux = y \text{ and}$

$$\frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = \frac{u+1}{1-u}$$

$$x \frac{du}{dx} = \frac{u+1}{1-u} - \frac{u(1-u)}{1-u}$$

$$= \frac{1+u^2}{1-u}$$

$$\frac{1-u}{1+u^2} dx = \frac{1}{x} dx$$

$$\left(\frac{1}{1+u^2} - \frac{u}{1+u^2} \right) du = \frac{1}{x} dx$$

$\uparrow$
 $w = 1+u^2$

$$dw = 2u du, \quad \frac{1}{2} dw = u du$$

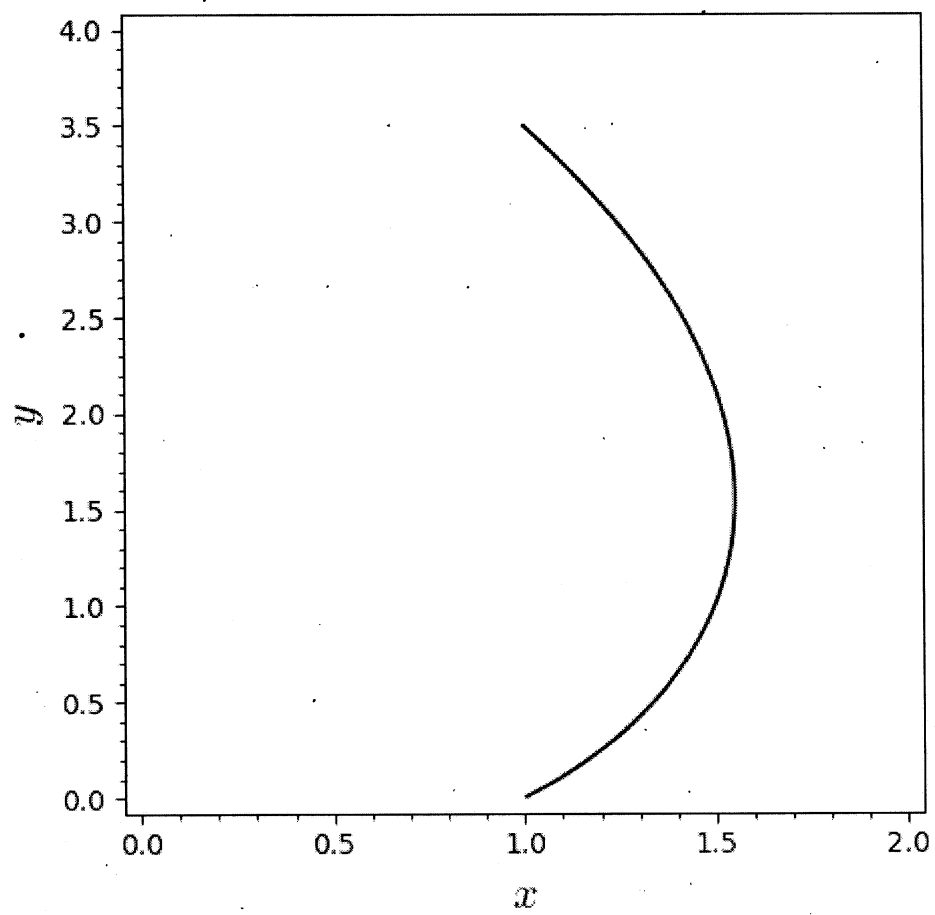
$$\tan^{-1}(u) - \frac{1}{2} \ln(1+u^2) = \ln|x| + C$$

$$y(1) = 0 \Rightarrow u(1) = 0 \Rightarrow C = 0$$

Result...

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x|$$

FYI, THE GRAPH OF THE
PATH IS
ATTACHED.



3.) a) $\frac{dy}{dx} = \frac{y-0}{x-\frac{x}{2}} = \frac{2y}{x}$

$$\boxed{\frac{dy}{dx} = \frac{2y}{x}}$$

b) Separable: $\frac{1}{y} dy = \frac{2}{x} dx$

$$\ln|y| = 2\ln|x| + C = \ln(x^2) + C,$$

$$|y| = C_2 x^2 \Rightarrow$$

$$\boxed{y(x) = Cx^2}$$

c) Linear: $\frac{dy}{dx} - \frac{2}{x}y = 0$

$$\mu(x) = e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$\frac{1}{x^2} y = \int 0 dx = C \Rightarrow \boxed{y(x) = Cx^2}$$

d) Homogeneous: $u = \frac{y}{x}, \quad y = ux$

$$\frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = 2u$$

$$\frac{1}{u} du = \frac{1}{x} dx$$

$$\ln|u| = \ln|x| + C,$$

$$u = C_2 x$$

$$\frac{y}{x} = C_2 x \Rightarrow$$

$$\boxed{y(x) = Cx^2}$$

$$4.) (x+y)y' = 1$$

$$\text{Let } u = x+y$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$u \left(\frac{du}{dx} - 1 \right) = 1 \Rightarrow u \frac{du}{dx} = 1+u$$

$$\frac{du}{dx} = \frac{1+u}{u}$$

$$\frac{u}{1+u} du = dx, \text{ Assuming } u \neq -1 \quad \begin{matrix} * \\ \text{SEE} \\ \text{BELOW} \end{matrix}$$

$$w = 1+u$$

$$dw = du$$

$$u = w-1$$

$$\frac{w-1}{w} dw = dx$$

$$w - \ln|w| = x + C_1$$

$$1+u - \ln|1+u| = x + C_1$$

$$1+x+y - \ln|1+x+y| = x + C_1$$

$$y - \ln|1+x+y| = C_2$$

$$y - \ln|1+x+y| = C$$

$$u = -1 \Rightarrow$$

$$y = -x-1,$$

WHICH IS A

SINGULAR

SOLUTION.

$$5.) \quad y' = \frac{x^2}{y^2} + \frac{y}{x}$$

$$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = \frac{1}{u^2} + u \Rightarrow x \frac{du}{dx} = \frac{1}{u^2} \Rightarrow u^2 du = \frac{1}{x} dx$$

$$\frac{1}{3} u^3 = \ln|x| + C_1$$

$$u^3 = 3 \ln|x| + C_2 \Rightarrow y^3 = 3x^3 \ln|x| + Cx^3$$

$$y(x) = \sqrt[3]{3x^3 \ln|x| + Cx^3}$$

$$= x \sqrt[3]{C + 3 \ln|x|}$$

$$6.) \quad x^2 y' + 2xy = 5y^4$$

$$y^{-4} y' + \frac{2}{x} y^{-3} = 5x^{-2}$$

$$u = y^{-3}, \text{ Assume } y \neq 0$$

$$\frac{du}{dx} = -3y^{-4} \frac{dy}{dx} \Rightarrow -\frac{1}{3} \frac{du}{dx} + \frac{2}{x} u = 5x^{-2}$$

$$\frac{du}{dx} - \frac{6}{x} u = -15x^{-2}$$

$$\mu(x) = e^{\int -\frac{6}{x} dx} = e^{-6 \ln|x|} = \frac{1}{x^6}$$

$$\frac{1}{x^6} u = \int \frac{-15}{x^8} dx = \frac{15}{7} x^{-7} + C$$

$$u = \frac{15}{7x} + Cx^6 \quad y^3 = \frac{1}{\frac{15}{7x} + Cx^6} = \frac{7x}{15 + C_2 x^7}$$

$$y(x) = \sqrt[3]{\frac{7x}{15 + Cx^7}}$$

$y(x) \equiv 0$ IS ALSO A SOLUTION.

7.) $y'' = y'e^y; \quad y(0) = 0, \quad y'(0) = 1$

$$y' = u, \quad y'' = u' = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$$

$$u \frac{du}{dy} = u e^y \Rightarrow \frac{du}{dy} = e^y \Rightarrow du = e^y dy$$

$$u = e^y + C$$

$$y' = e^y + C$$

$$y'(0) = 1 \text{ AND } y(0) = 0$$

↓

$$1 = e^0 + C \Rightarrow C = 0$$

$$y' = e^y \Rightarrow e^{-y} dy = dx$$

$$-e^{-y} = x + D$$

$$y(0) = 0 \Rightarrow D = -1$$

$$e^{-y} = 1 - x$$

$$-y = \ln(1-x)$$

$$y(x) = -\ln(1-x)$$

8.) $xy'' + y' = 4x$

$$y' = u, \quad y'' = u'$$

$$u' + \frac{1}{x}u = 4$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x \text{ Assuming } x > 0$$

$$xu = \int 4x dx = 2x^2 + C$$

$$u = 2x + \frac{C}{x}$$

$$\frac{dy}{dx} = 2x + \frac{C}{x} \Rightarrow y = x^2 + C \ln x + D$$

$$y(x) = x^2 + C \ln x + D, \quad x > 0$$

9.) $x^2 y'' - 4xy' + 6y = 0$

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$$

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$$P(x) = -\frac{4}{x}$$

$$Q(x) = \frac{6}{x^2}$$

} THE COEFFICIENT FUNCTIONS ARE NOT CONTINUOUS (IN FACT, NOT DEFINED)

AT $x = 0$.

Our THEOREM ONLY GUARANTEES A

UNIQUE SOLUTION WHERE THE

COEFFICIENT FUNCTIONS ARE CONTINUOUS.

10.) $y_1(x) = x^2 + 1$

$$y_2(x) = x^2 + 3x$$

$$y_3(x) = 1 - 3x$$

ARE LINEARLY DEPENDENT BECAUSE

$$y_1(x) = y_2(x) + y_3(x).$$

11.) $yy'' + (y')^2 = 0$, $y_1(x) = 1$, $y_2(x) = \sqrt{x}$

a) $y_1(x) = 1$, $y_1'(x) = 0$, $y_1''(x) = 0 \Rightarrow (1)(0) + (0)^2 = 0 \checkmark$

$$y_2(x) = x^{1/2}, y_2'(x) = \frac{1}{2}x^{-1/2}, y_2''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow (x^{1/2})\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2 = -\frac{1}{4x} + \frac{1}{4x} = 0 \checkmark$$

b) $y(x) = 1 + x^{1/2}$, $y'(x) = \frac{1}{2}x^{-1/2}$, $y''(x) = -\frac{1}{4}x^{-3/2}$

$$(1 + x^{1/2})\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)^2 = -\frac{1}{4}x^{-3/2} - \frac{1}{4x} + \frac{1}{4x} = -\frac{1}{4}x^{-3/2} \neq 0 \checkmark$$

c) THE EQUATION IS NOT LINEAR. WE SHOULD NOT

EXPECT A SUM OF SOLUTIONS TO BE A SOLUTION.