

(1)  $x'' + 9x = 0; \quad x(0) = 1, \quad x'(0) = \sqrt{3}$

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow \alpha = 0, \beta = 3$$

$$x(t) = c_1 \cos 3t + c_2 \sin 3t$$

$$x(0) = 1 \Rightarrow c_1 = 1, \quad x'(0) = \sqrt{3} \Rightarrow 3c_2 = \sqrt{3} \Rightarrow c_2 = \frac{\sqrt{3}}{3}$$

$$A = \sqrt{(1)^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{\frac{12}{9}} = \frac{2}{3}\sqrt{3}$$

$$c_1, c_2 > 0 \Rightarrow \varphi \text{ in QUAD I} \Rightarrow \varphi = \tan^{-1} \frac{3}{\sqrt{3}} \Rightarrow \varphi = \frac{\pi}{3}$$

$$x(t) = \frac{2}{\sqrt{3}} \sin\left(3t + \frac{\pi}{3}\right)$$

$$x'(t) = 2\sqrt{3} \cos\left(3t + \frac{\pi}{3}\right)$$

$$2\sqrt{3} \cos\left(3t + \frac{\pi}{3}\right) = -3$$

$$\Rightarrow \cos\left(3t + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow 3t + \frac{\pi}{3} = \frac{5\pi}{6} \pm 2\pi k \quad \text{or} \quad \frac{7\pi}{6} \pm 2\pi k$$

$$\Rightarrow 3t = \frac{\pi}{2} \pm 2\pi k \quad \text{or} \quad \frac{5\pi}{6} \pm 2\pi k$$

$$\Rightarrow t = \frac{\pi}{6} \pm \frac{2}{3}\pi k \quad \text{or} \quad \frac{5\pi}{18} \pm \frac{2}{3}\pi k$$

For ANY integer  $k$

Obviously, only interested  
in  $t \geq 0$ .

(2)

$$(2) \quad x'' + 10x' + 16x = 0; \quad x(0) = -1, \quad x'(0) = 0$$

$$r^2 + 10r + 16 = 0$$

$$(r+2)(r+8) = 0 \Rightarrow r = -2, r = -8$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-8t}$$

$$x(0) = -1 \Rightarrow c_1 + c_2 = -1$$

$$x'(0) = 0 \Rightarrow -2c_1 - 8c_2 = 0$$

$$-6c_2 = -2 \Rightarrow c_2 = \frac{1}{3} \Rightarrow c_1 = -\frac{4}{3}$$

$$x(t) = -\frac{4}{3} e^{-2t} + \frac{1}{3} e^{-8t}$$

THE SYSTEM IS OVERDAMPED--

NO OSCILLATIONS AND

$$b^2 - 4mk = 100 - 64 > 0.$$

$$(3) \quad 9x'' + 6x' + 37x = 0; \quad x(0) = 1, \quad x'(0) = -2$$

$$9r^2 + 6r + 1 = -36$$

$$(3r+1)^2 = -36 \Rightarrow 3r = -1 \pm 6i \Rightarrow r = -\frac{1}{3} \pm 2i$$

$$x(t) = c_1 e^{-\frac{1}{3}t} \cos 2t + c_2 e^{-\frac{1}{3}t} \sin 2t$$

$$x(0) = 1 \Rightarrow c_1 = 1$$

$$x'(0) = -2 \Rightarrow -\frac{1}{3}c_1 + 2c_2 = -2 \Rightarrow 2c_2 = -\frac{5}{3} \Rightarrow c_2 = -\frac{5}{6}$$

$$x(t) = e^{-\frac{1}{3}t} \left( \cos 2t - \frac{5}{6} \sin 2t \right)$$

$$A = \sqrt{(1)^2 + \left(-\frac{5}{6}\right)^2} = \sqrt{1 + \frac{25}{36}} = \frac{\sqrt{61}}{6}$$

$$c_1 > 0 \text{ \& } c_2 < 0 \Rightarrow \varphi \text{ IN QUAD II} \Rightarrow \varphi = \tan^{-1}\left(-\frac{6}{5}\right) + \pi$$

$$x(t) = \frac{\sqrt{61}}{6} e^{-\frac{1}{3}t} \sin \left( 2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right) \right)$$

Now need 2<sup>nd</sup> pos t FOR WHICH

$$2t + \pi + \tan^{-1}\left(-\frac{6}{5}\right) = k\pi$$

$$t = \frac{k\pi - \pi - \tan^{-1}\left(-\frac{6}{5}\right)}{2}$$

$$k=0 \Rightarrow t < 0$$

$$k=1 \Rightarrow t \approx 0.438 \text{ sec.}$$

$$k=2 \Rightarrow t \approx 2.009 \text{ sec}$$

(4)

$$\textcircled{4} \quad \frac{1}{4}x'' + 2x' + 4x = 0; \quad x(0) = 0, \quad x'(0) = -3$$

$$\frac{1}{4}r^2 + 2r + 4 = 0 \Rightarrow r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0$$

$$\Rightarrow r = -4, r = -4$$

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$x(0) = 0 \Rightarrow c_1 = 0$$

$$x'(t) = c_2 e^{-4t} - 4c_2 t e^{-4t}$$

$$x'(0) = -3 = c_2 = -3$$

$$x(t) = -3t e^{-4t}$$

THE SYSTEM IS CRITICALLY DAMPED.

No oscillations AND  $b^2 - 4mk = 0$

(5)

$$(5) \quad 0.1x'' + 0.4x' + 2x = 0; \quad x(0) = -1, \quad x'(0) = 0$$

$$r^2 + 4r + 20 = 0$$

$$r^2 + 4r + 4 = -16 \Rightarrow (r+2)^2 = -16 \Rightarrow r = -2 \pm 4i$$

$$\alpha = -2, \beta = 4$$

$$x(t) = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t$$

$$x(0) = -1 \Rightarrow c_1 = -1$$

$$x'(0) = 0 \Rightarrow -2c_1 + 4c_2 = 0 \Rightarrow c_2 = -\frac{1}{2}$$

$$x(t) = -e^{-2t} \cos 4t - \frac{1}{2} e^{-2t} \sin 4t$$

$$A = \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$c_1 < 0 \text{ AND } c_2 < 0 \Rightarrow \phi \text{ IS IN QUAD III} \Rightarrow \phi = \tan^{-1}(2) + \pi$$

$$x(t) = \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + \pi + \tan^{-1}(2))$$

Now we need 1<sup>st</sup> pos t for which

$$4t + \pi + \tan^{-1}(2) = k\pi$$

$$t = \frac{k\pi - \pi - \tan^{-1}(2)}{4}$$

$$k = 2 \Rightarrow t \approx 0.5086 \text{ sec}$$

(6)

$$(6) \quad y'' + 8y = 2e^{-x}$$

Homo. eqn:  $y'' + 8y = 0$

$$r^2 + 8 = 0 \Rightarrow r = \pm \sqrt{8}i = \pm 2\sqrt{2}i \quad \alpha = 0, \beta = 2\sqrt{2}$$

$$y_c(x) = c_1 \cos(2\sqrt{2}x) + c_2 \sin(2\sqrt{2}x)$$

Non HOMO. Eqn. HAS  $g(x) = 2e^{-x}$

$$y_p(x) = x^s (Ae^{-x}) \quad \text{WITH } s = 0$$

$$y_p(x) = Ae^{-x}$$

$$y_p'(x) = -Ae^{-x}$$

$$y_p''(x) = Ae^{-x}$$

$$Ae^{-x} + 8Ae^{-x} = 2e^{-x} \Rightarrow 9A = 2$$

$$A = \frac{2}{9}$$

$$y_p(x) = \frac{2}{9}e^{-x}$$

$$y(x) = c_1 \cos(2\sqrt{2}x) + c_2 \sin(2\sqrt{2}x) + \frac{2}{9}e^{-x}$$

$$\textcircled{7} \quad y'' + 3y' + 2y = 4x^2$$

Homo Eqn:  $y'' + 3y' + 2y = 0$

$$r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1, r = -2$$

$$y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$$

NonHomo. Eqn HAS  $g(x) = 4x^2$ .

$$y_p(x) = x^s (Ax^2 + Bx + C) \text{ WITH } s = 0$$

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$$

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = 4x^2$$

$$2A = 4$$

$$A = 2, B = -6, C = 7$$

$$6A + 2B = 0$$

$$2A + 3B + 2C = 0$$

$$y_p(x) = 2x^2 - 6x + 7$$

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + 2x^2 - 6x + 7$$

(8)

$$\textcircled{8} \quad y'' - 3y' = 4 \sin x$$

Homo. Eqn:  $y'' - 3y' = 0$

$$r^2 - 3r = 0 \Rightarrow r(r-3) = 0 \Rightarrow r=0, r=3$$

$$y_c(x) = c_1 + c_2 e^{3x}$$

NonHomo Eqn HAS  $g(x) = 4 \sin x$ .

$$y_p(x) = x^s (A \cos x + B \sin x) \quad \text{WITH } s=0$$

$$y_p(x) = A \cos x + B \sin x$$

$$y_p'(x) = -A \sin x + B \cos x$$

$$y_p''(x) = -A \cos x - B \sin x$$

$$-A \cos x - B \sin x - 3(-A \sin x + B \cos x) = 4 \sin x$$

$$-A - 3B = 0$$

$$-B + 3A = 4$$

$$-10B = 4$$

$$B = -\frac{2}{5}$$

$$\Rightarrow A = \frac{6}{5}$$

$$y_p(x) = \frac{6}{5} \cos x - \frac{2}{5} \sin x$$

$$y(x) = c_1 + c_2 e^{3x} + \frac{6}{5} \cos x - \frac{2}{5} \sin x$$