

①

MTH 240 Assignment 7 Key

① $y'' + 2y' + y = 2x + 3 - e^5 x$

Homogeneous Eqn: $y'' + 2y' + y = 0$

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0, r = -1, r = -1$$

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Non Homogeneous Eqn has $g(x) = 2x + 3 - e^5 x$

$$y_p(x) = x^s (Ax + B) \text{ with } s = 0$$

$$y_p(x) = Ax + B$$

$$y_p'(x) = A$$

$$y_p''(x) = 0$$

$$2A + Ax + B = 2x + 3 - e^5 x$$

$$A = 2 - e^5$$

$$2A + B = 3 \Rightarrow B = 3 - 2(2 - e^5) \\ = -1 + 2e^5$$

$$y_p(x) = (2 - e^5)x - 1 + 2e^5$$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + (2 - e^5)x - 1 + 2e^5$$

(2)

$$(2) \quad 2y'' + 6y' - 20y = 60 \sin 2x$$

Homo Eqn: $2y'' + 6y' - 20y = 0$

$$y'' + 3y' - 10y = 0$$

$$r^2 + 3r - 10 = 0 \Rightarrow (r+5)(r-2) = 0$$

$$r = -5, r = 2$$

$$y_c(x) = c_1 e^{-5x} + c_2 e^{2x}$$

Nonhomo. Eqn HAS $g(x) = 60 \sin 2x$ (or let's use $30 \sin 2x$)

$$y_p(x) = x^s (A \cos 2x + B \sin 2x) \text{ with } s = 0$$

$$y_p'(x) = -2A \sin 2x + 2B \cos 2x$$

$$y_p''(x) = -4A \cos 2x - 4B \sin 2x$$

$$-4A \cos 2x - 4B \sin 2x$$

$$6B \cos 2x - 6A \sin 2x$$

$$-10A \cos 2x - 10B \sin 2x = 30 \sin 2x$$

$$\left. \begin{array}{l} 6B - 14A = 0 \\ -14B - 6A = 30 \end{array} \right\} B = \frac{7}{3}A$$

$$-\frac{14 \cdot 7}{3}A - \frac{18}{3}A = 30 \Rightarrow A = \frac{-90}{116} = \frac{-45}{58}$$

$$B = -\frac{105}{58}$$

$$y_p(x) = -\frac{45}{58} \cos 2x - \frac{105}{58} \sin 2x$$

$$y(x) = c_1 e^{-5x} + c_2 e^{2x} - \frac{45}{58} \cos 2x - \frac{105}{58} \sin 2x$$

(3)

$$(3) \quad y'' - 5y' + 4y = 2e^{4x}; \quad y(0) = 1, \quad y'(0) = -1$$

Homog. eqn: $y'' - 5y' + 4y = 0$

$$r^2 - 5r + 4 = 0 \Rightarrow (r-4)(r-1) = 0 \Rightarrow r=4, r=1$$

$$y_c(x) = c_1 e^{4x} + c_2 e^x$$

NonHomog eqn has $g(x) = 2e^{4x} \Rightarrow y_p(x) = x^s (Ae^{4x})$ with $s=1$

$$y_p(x) = Axe^{4x}$$

$$y_p'(x) = Ae^{4x} + 4Axe^{4x}$$

$$y_p''(x) = 8Ae^{4x} + 16Axe^{4x}$$

$$16Axe^{4x} + 8Ae^{4x} - 5Ae^{4x} - 20Axe^{4x} + 4Axe^{4x} = 3Ae^{4x} = 2e^{4x} \Rightarrow A = \frac{2}{3}$$

$$y_p(x) = \frac{2}{3}xe^{4x}$$

$$y(x) = c_1 e^{4x} + c_2 e^x + \frac{2}{3}xe^{4x}$$

$$y(0) = 1 \Rightarrow c_1 + c_2 = 1$$

$$y'(0) = -1 \Rightarrow 4c_1 + c_2 + \frac{2}{3} = -1$$

$$3c_1 + \frac{2}{3} = -2 \Rightarrow c_1 = -\frac{8}{9}$$

$$c_2 = \frac{17}{9}$$

$$y(x) = -\frac{8}{9}e^{4x} + \frac{17}{9}e^x + \frac{2}{3}xe^{4x}$$

④ $y^{(3)} + y'' = x + e^{-x}; \quad y(0)=1, y'(0)=0, y''(0)=1$

Homo. eqn: $y^{(3)} + y'' = 0$

$r^3 + r^2 = 0 \Rightarrow r^2(r+1) = 0 \Rightarrow r=0, r=0, r=-1$

$y_c(x) = c_1 + c_2 x + c_3 e^{-x}$

Non Homo HAS $g(x) = x + e^{-x} \Rightarrow y_p(x) = x^s (Ax+B) + x^t C e^{-x}$
 $s=2 \quad t=1$

$y_p(x) = Ax^3 + Bx^2 + Cx e^{-x}$

$y_p'(x) = 3Ax^2 + 2Bx + Ce^{-x} - Cx e^{-x}$

$y_p''(x) = 6Ax + 2B - 2Ce^{-x} + Cx e^{-x}$

$y_p^{(3)}(x) = 6A + 3C e^{-x} - Cx e^{-x}$

$y_p^{(3)} + y_p'' = 6Ax + (2B+6A) + Ce^{-x} = x + e^{-x}$

$6A = 1$

$2B+6A = 0$

$C = 1$

$A = \frac{1}{6}$

$B = -\frac{1}{2}$

$y_p(x) = \frac{1}{6} x^3 - \frac{1}{2} x^2 + x e^{-x}$

$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{1}{6} x^3 - \frac{1}{2} x^2 + x e^{-x}$

$y(0)=1 \Rightarrow c_1 + c_3 = 1$

$y'(x) = c_2 - c_3 e^{-x} + \frac{1}{2} x^2 - x + e^{-x} - x e^{-x}$

$y'(0)=0 \Rightarrow c_2 - c_3 + 1 = 0$

$y''(x) = c_3 e^{-x} + x - 1 - 2e^{-x} + x e^{-x}$

$y''(0)=1 \Rightarrow c_3 - 1 - 2 = 1$

$c_1 + c_3 = 1$

$c_2 - c_3 = -1$

$c_3 = 4$

$c_2 = 3$

$c_1 = -3$

$y(x) = -3 + 3x + 4e^{-x} + \frac{1}{6} x^3 - \frac{1}{2} x^2 + x e^{-x}$

(5)

$$\textcircled{5} \quad y'' - 6y' + 9y = 5t^6 e^{3t}$$

Homo eqn: $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r=3, r=3$$

$$y_c(t) = c_1 e^{3t} + c_2 t e^{3t}$$

NonHomo eqn HAS $g(t) = 5t^6 e^{3t} \Rightarrow y_p(t) = t^s (At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G) e^{3t}$

WITH $s=2$

$$y_p(t) = (At^8 + Bt^7 + Ct^6 + Dt^5 + Et^4 + Ft^3 + Gt^2) e^{3t}$$

$$\textcircled{6} \quad y'' + y = \sin t + t \cos t + e^t$$

Homo eqn: $y'' + y = 0$

$$r^2 + 1 = 0 \quad r = \pm i \Rightarrow y_c(t) = c_1 \cos t + c_2 \sin t$$

NonHomo #1 HAS $g(t) = \sin t$

$$\Rightarrow y_{p1}(t) = t^s (A \cos t + B \sin t) \quad \text{WITH } s=1$$

NonHomo #2 HAS $g(t) = t \cos t$

$$\Rightarrow y_{p2}(t) = t^s [(Ct+D) \cos t + (Et+F) \sin t] \quad \text{WITH } s=2$$

NonHomo #3 HAS $g(t) = e^t$

$$\Rightarrow y_{p3}(t) = t^s (Ge^t) \quad \text{WITH } s=0$$

$$y_p(t) = At \cos t + Bt \sin t + (Ct^3 + Dt^2) \cos t + (Et^3 + Ft^2) \sin t + Ge^t$$

(6)

$$(7) \quad x'' - 4x' + 4x = \frac{e^{2t}}{t^2}, \quad t > 0$$

Homo eqn: $x'' - 4x' + 4x = 0$

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r=2, r=2$$

$$x_c(t) = c_1 e^{2t} + c_2 t e^{2t}; \quad x_1(t) = e^{2t}, \quad x_2(t) = t e^{2t}$$

VARIATION OF PARAMETERS

$$W = \begin{vmatrix} e^{2t} & t e^{2t} \\ 2e^{2t} & e^{2t} + 2t e^{2t} \end{vmatrix} = e^{4t}$$

$$y_1(t) = \int \frac{-t e^{2t} \left(\frac{e^{2t}}{t^2} \right)}{e^{4t}} dt = \int -\frac{1}{t} dt = -\ln t$$

$$y_2(t) = \int \frac{e^{2t} \left(\frac{e^{2t}}{t^2} \right)}{e^{4t}} dt = \int \frac{1}{t^2} dt = -\frac{1}{t}$$

$$x_p(t) = -e^{2t} \ln t - e^{2t}$$

$$x(t) = c_1 e^{2t} + c_2 t e^{2t} - e^{2t} \ln t, \quad t > 0$$

(7)

⑧ $y'' + y = \csc^2 x$

Homo. eqn: $y'' + y = 0$

$$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_c(x) = c_1 \cos x + c_2 \sin x$$

$$y_1(x) = \cos x, \quad y_2(x) = \sin x$$

VARIATION OF PARAMETERS...

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$V_1(x) = \int \frac{-\sin x \csc^2 x}{1} dx$$

$$= \int -\csc x dx = \ln |\csc x + \cot x|$$

$$V_2(x) = \int \frac{\cos x \csc^2 x}{1} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{u^2} du = -\frac{1}{u} = -\csc x$$

$$u = \sin x$$

$$du = \cos x dx$$

$$y_p(x) = \cos x \ln |\csc x + \cot x| - 1$$

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln |\csc x + \cot x| - 1$$

9) $(1-x)y'' + xy' - y = (1-x)^2, \quad x > 1$

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = (1-x)$$

a) $y_1(x) = x, \quad y_1'(x) = 1, \quad y_1''(x) = 0$

$$\frac{x}{1-x} \cdot 1 - \frac{1}{1-x} \cdot x = 0 \Rightarrow y_1(x) = x \text{ solves Homo. Eqn.}$$

$$y_2(x) = e^x, \quad y_2'(x) = e^x, \quad y_2''(x) = e^x$$

$$e^x + \frac{x e^x}{1-x} - \frac{e^x}{1-x} = e^x + \frac{e^x(x-1)}{1-x} = e^x - e^x = 0$$

$\Rightarrow y_2(x) = e^x$ solves Homo. Eqn.

$$y_c(x) = C_1 x + C_2 e^x$$

b) $W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = x e^x - e^x = (x-1)e^x > 0 \text{ when } x > 1$

x & e^x are lin. indep. on $(1, \infty)$.

c) $y_1(x) = \int \frac{-(1-x)e^x}{(x-1)e^x} dx = \int 1 dx = x$

$$y_2(x) = \int \frac{(1-x)x}{(x-1)e^x} dx = \int -x e^{-x} dx = x e^{-x} - \int e^{-x} dx = x e^{-x} + e^{-x}$$

$u = -x \quad du = -dx$
 $dv = e^{-x} dx \quad v = -e^{-x}$

$$y_p(x) = x^2 + x + 1$$

Problem 9 CONTINUED...

$$d) \quad y(x) = c_1 x + c_2 e^x + x^2 + x + 1$$

$$e) \quad y(a) = 1 \Rightarrow 2c_1 + c_2 e^a + 7 = 1$$

$$y'(x) = c_1 + c_2 e^x + 2x + 1$$

$$y'(a) = -4 \Rightarrow c_1 + c_2 e^a + 5 = -4$$

$$2c_1 + c_2 e^a = -6$$

$$c_1 + c_2 e^a = -9$$

$$c_1 = 3$$

$$c_2 = -12 e^{-a}$$

$$y(x) = x - 12 e^{x-a} + x^2 + x + 1$$