Math 240 - Test 1

September 10, 2026

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand (showing work), unless otherwise specified.

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1. (8 points) Consider the differential equation $\dfrac{dy}{dx} = y^2$.

   (a) State the order of the equation and say which variable is independent and which is dependent.

   1ST ORDER EQUATION, y IS DEPENDENT, X IS INDEPENDENT.

   (b) Say whether the equation is linear or nonlinear and explain how you know.

   NONLINEAR BECAUSE OF y^2.

   (c) Verify by substitution that $y(x) = \dfrac{1}{C-x}$ is a solution for any arbitrary constant C.

   $$\frac{d}{dx} y = \frac{d}{dx} \frac{1}{C-x} = \frac{d}{dx} (C-x)^{-1} = -1 (C-x)^{-2}(-1) = \frac{1}{(C-x)^2} = y^2 \checkmark$$

   (d) Use the solution from part (c) to find the particular solution that satisfies $y(2) = 7$.

   $$y(2) = 7 \Rightarrow \frac{1}{C-2} = 7 \Rightarrow C-2 = \frac{1}{7} \Rightarrow C = \frac{15}{7}$$

   $$\boxed{y(x) = \frac{7}{15-7x}}$$

   (e) Find (by observation) a singular solution of the differential equation.

   $y(x) \equiv 0$ (CONSTANT FUNCTION)

2. (2 points) In modelling a problem situation, a researcher proposed the following differential equation:

   $$\frac{d^2u}{dx^2} = 3x\frac{d^2u}{dy^2}.$$

   Explain why this differential equation is a partial differential equation despite the fact that the researcher wrote the equation using the notation of ordinary derivatives.

   THERE ARE TWO INDEPENDENT VARIABLES: X & y.

   u DEPENDS ON X & y.

3. (3 points) The graph of $y = f(x)$ has the following property:

The tangent line at the point (x, y) passes through the point $(-y, x)$.

Write a differential equation that describes the family of such curves.

$$\frac{dy}{dx} = \underbrace{\frac{y-x}{x-(-y)}}_{\text{SLOPE THROUGH } (x,y) \text{ AND } (-y,x).} = \frac{y-x}{x+y} \qquad \boxed{\frac{dy}{dx} = \frac{y-x}{x+y}}$$

4. (12 points) Using our existence/uniqueness theorems, analyze each initial value problem to determine which one of these applies. **Do not attempt to solve the equations.**

(A) A solution exists, but it is not guaranteed to be unique.
(B) There is a unique solution.
(C) A solution is not guaranteed to exist.

Be sure to show work or explain.

(a) $(x^2+1)\dfrac{dy}{dx} + 3xy = 6x, \quad y(1) = 7$

(B)

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

LINEAR, $p(x) = \dfrac{3x}{x^2+1}$, $q(x) = \dfrac{6x}{x^2+1}$

p & q ARE CONTINUOUS EVERYWHERE. THERE IS A UNIQUE SOLUTION THROUGH ANY INITIAL PT.

(b) $\dfrac{dy}{dx} = (x-3)(y+1)^{2/3}, \quad y(0) = -1$

$f(x,y) = (x-3)(y+1)^{2/3}$ IS CONTINUOUS EVERYWHERE.

(A)

$f_y(x,y) = \frac{2}{3}(x-3)(y+1)^{-1/3}$ IS NOT DEFINED WHEN $y=-1$

(c) $\dfrac{dy}{dx} = \sqrt{x-y}, \quad y(2) = 2$

$f(x,y) = \sqrt{x-y}$ IS NOT CONTINUOUS ON A RECTANGLE CONTAINING $(2,2)$ IN ITS INTERIOR.

(C)

5. (6 points) An object gradually speeds up from rest in such a way that its acceleration over the first 10 seconds is given by

$$\frac{d^2x}{dt^2} = 0.12t^2 + 0.6t; \quad x(0) = 0,\ x'(0) = 0,$$

where x is measured in feet. Find $x(5)$ and $x'(5)$.

$\frac{dx}{dt} = 0.04t^3 + 0.3t^2 + C, \quad x'(0) = 0 \Rightarrow C = 0$

$x'(t) = 0.04t^3 + 0.3t^2$

$\boxed{x'(5) = 12.5 \text{ ft/s}}$

$x(t) = 0.01t^4 + 0.1t^3 + C, \quad x(0) = 0 \Rightarrow C = 0$

$x(t) = 0.01t^4 + 0.1t^3$

$\boxed{x(5) = 18.75 \text{ ft}}$

6. (12 points) Solve the initial value problem: $x\frac{dy}{dx} - y = 2x^2y, \quad y(1) = 1.$

$\frac{dy}{dx} = \frac{2x^2y + y}{x}$

$\frac{dy}{dx} = \frac{(2x^2+1)y}{x}$

$\frac{1}{y}\,dy = \frac{2x^2+1}{x}\,dx$

$\int \frac{1}{y}\,dy = \int \left(2x + \frac{1}{x}\right) dx$

$\ln|y| = x^2 + \ln|x| + C_1$

$e^{\ln|y|} = e^{x^2 + \ln|x| + C_1}$

$|y| = e^{x^2} e^{\ln|x|} e^{C_1}$

$|y| = C_2 |x| e^{x^2}$

$y(x) = C_3\, x e^{x^2}$

$y(1) = 1 \Rightarrow C_3 e = 1 \Rightarrow C_3 = \frac{1}{e}$

$\boxed{y(x) = x e^{x^2 - 1}}$

7. (12 points) Solve the initial value problem: $\frac{dy}{dx} - y = e^{-2x}$, $y(0) = 1$.

$\mu(x) = e^{\int -dx} = e^{-x}$

$e^{-x} y = \int e^{-x} e^{-2x}\, dx$

$= \int e^{-3x} \cdot dx$

$= -\frac{1}{3} e^{-3x} + C$

$y(x) = -\frac{1}{3} e^{-2x} + Ce^{x}$

$y(0) = 1 \Rightarrow -\frac{1}{3} + C = 1$

$\Rightarrow C = \frac{4}{3}$

$\boxed{y(x) = -\frac{1}{3} e^{-2x} + \frac{4}{3} e^{x}}$

$f(x,y) = y + e^{-2x}$

8. (8 points) Use Euler's method with $h = 0.1$ to estimate $y(0.3)$.

$\frac{dy}{dx} - y = e^{-2x}$, $y(0) = 1$

$y_{n+1} = y_n + 0.1\, f(x_n, y_n)$

$y_{n+1} = y_n + 0.1 \left(y_n + e^{-2x_n} \right)$

$y_0 = 1$, $x_0 = 0$

$y_1 = 1 + 0.1 \left(1 + 1 \right) = 1.2$

$x_1 = 0.1$

$y_2 = 1.2 + 0.1 \left(1.2 + e^{-0.2} \right)$

$\approx 1.401873075\ldots$

$x_2 = 0.2$

$y_3 = y_2 + 0.1 \left(y_2 + e^{-0.4} \right)$

$\approx 1.609092387\ldots$

$\boxed{y(0.3) \approx 1.609}$

9. (12 points) The acceleration of a certain sports car is proportional to the difference between its velocity and 250 km/h. The car can accelerate from rest to 100 km/h in 10 seconds (10/3600 hours). How long will it take for the car to accelerate from rest to 200 km/h?

$\frac{dv}{dt} = k(v-250)$, $v(0)=0$, $v\left(\frac{1}{360}\right) = 100$ (WORKING IN km & h)

$\frac{1}{v-250}\,dv = k\,dt$

$\ln|v-250| = kt + C_1$

$|v-250| = e^{kt+C_1}$

$= C_2 e^{kt}$

$v - 250 = C_3 e^{kt}$

$v(t) = 250 + C_3 e^{kt}$

$v(0) = 0 \Rightarrow C_3 = -250$

$v(t) = 250 - 250e^{kt}$

$v\left(\frac{1}{360}\right) = 100$

$\Rightarrow -150 = -250 e^{k/360}$

$k = 360 \ln\left(\frac{15}{25}\right)$

SOLVE

$v(t) = 200$

$-50 = -250e^{kt}$

$\ln\left(\frac{5}{25}\right) = kt$

$\boxed{t = \frac{\ln\left(\frac{1}{5}\right)}{360\ln\left(\frac{3}{5}\right)} \approx 0.00875 \text{ HOURS} \approx 31.5 \text{ SEC}}$

10. (11 points) Show that the equation IS NOT exact. Then determine the type of equation and find the general solution.

$$\underbrace{(y-12x)}_{M}\,dx + \underbrace{3x}_{N}\,dy = 0$$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 3 \Rightarrow$ NOT EXACT!

$3x\frac{dy}{dx} = 12x - y$

$\frac{dy}{dx} + \frac{1}{3x}y = 4$ } IT'S LINEAR!

$\mu(x) = e^{\int \frac{1}{3x}dx}$

$= e^{\frac{1}{3}\ln|x|} = |x|^{1/3}$

$= \sqrt[3]{x}$, $x > 0$

$\sqrt[3]{x}\,y = \int 4\sqrt[3]{x}\,dx$

$\sqrt[3]{x}\,y = 3x^{4/3} + C$

$\boxed{y(x) = 3x + Cx^{-1/3}}$

11. (10 points) Show that the equation is exact. Then find the general solution.

$$\underbrace{\left(\frac{1}{x} + 2y^2x\right)}_{m} dx + \underbrace{(2yx^2 - \cos y)}_{N} dy = 0$$

$\frac{\partial m}{\partial y} = 4yx = \frac{\partial N}{\partial x} \Rightarrow$ EXACT!

$F_x = \frac{1}{x} + 2y^2x$

$\Rightarrow F(x,y) = \ln|x| + y^2x^2 + g(y)$

$F_y = 2yx^2 - \cos y$

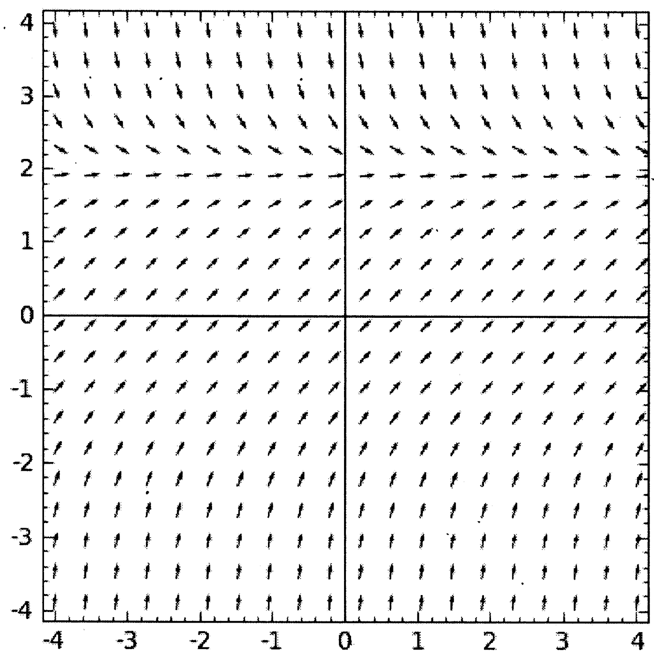
$\Rightarrow F(x,y) = y^2x^2 - \sin y + h(x)$

$F(x,y) = \ln|x| + y^2x^2 - \sin y$

SOLUTION:

$\ln|x| + y^2x^2 - \sin y = C$

12. (4 points) A direction field for $\frac{dy}{dx} = 1 - \frac{y^3}{8}$ is shown below. Use the direction field to solve the following problems.



(a) What is the unique solution passing through $(-2, 2)$?

IT LOOKS LIKE $\boxed{y(x) \equiv 2}$ (CONSTANT FUNC). THIS CHECKS OUT IN THE EQUATION!

(b) Suppose you have found a solution, $y(x)$, passing through a given point. What can you say about $\lim_{x\to\infty} y(x)$?

IT LOOKS LIKE $\boxed{\lim_{x\to\infty} y(x) = 2}$ REGARDLESS OF THE INITIAL POINT.